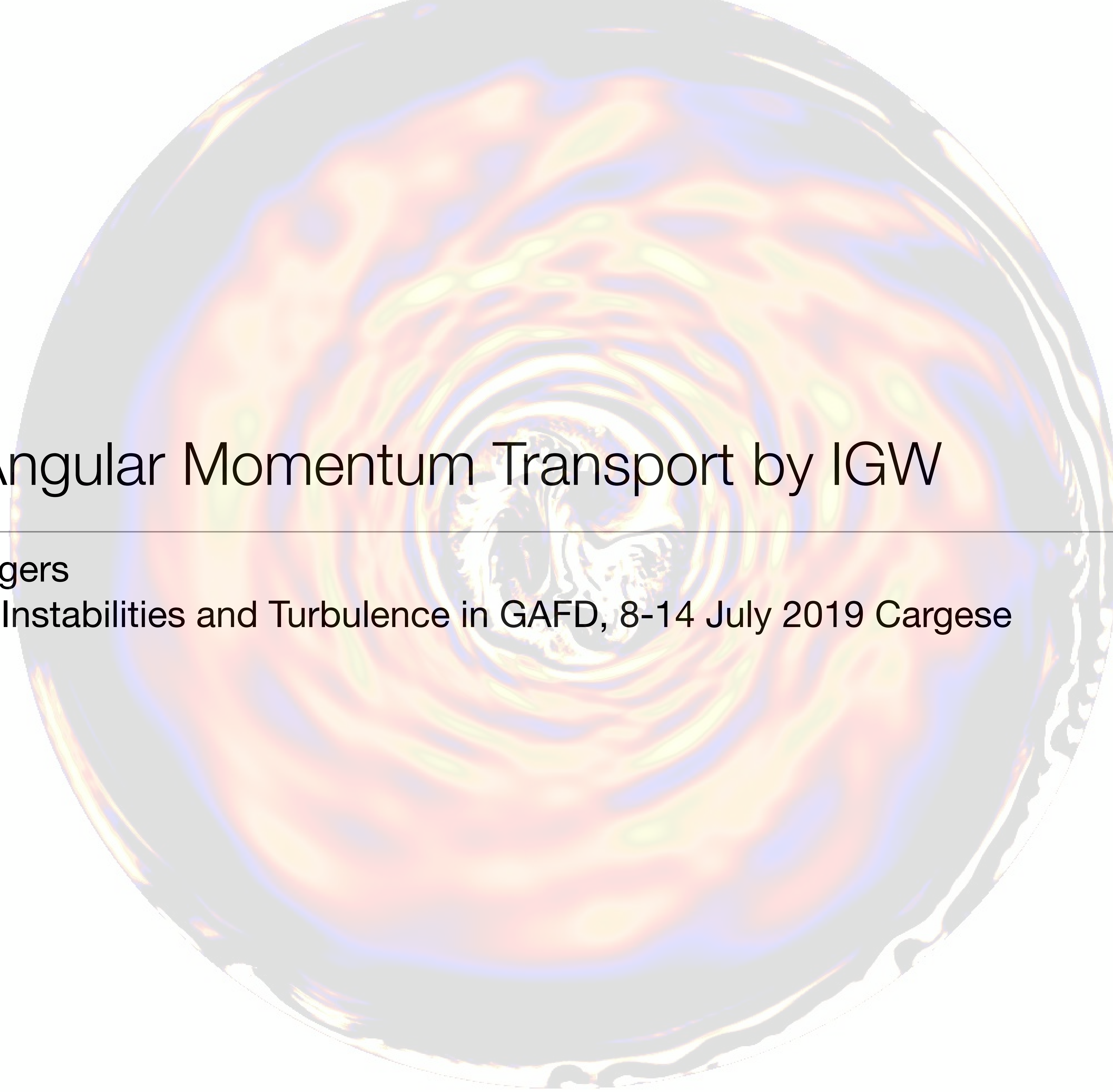
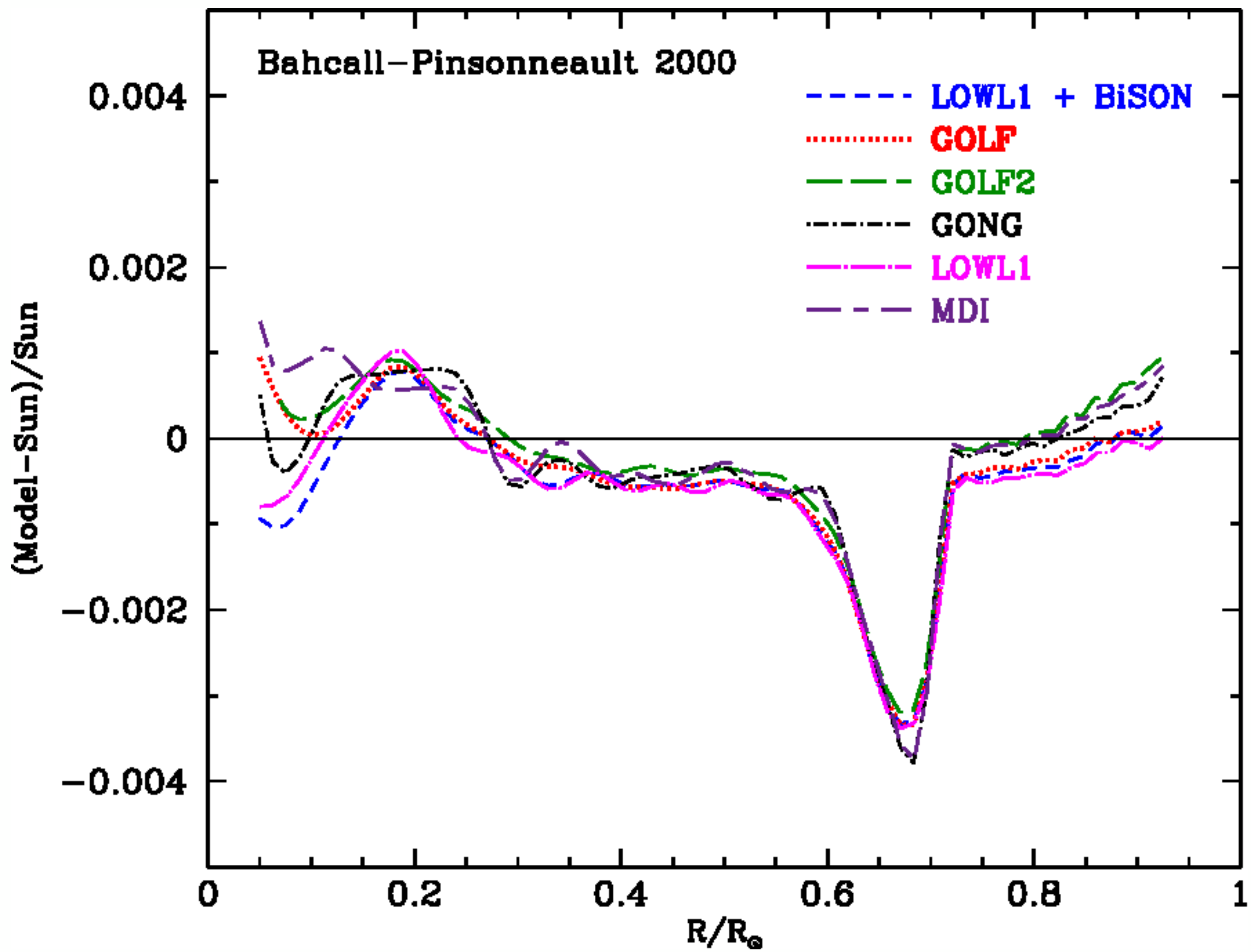


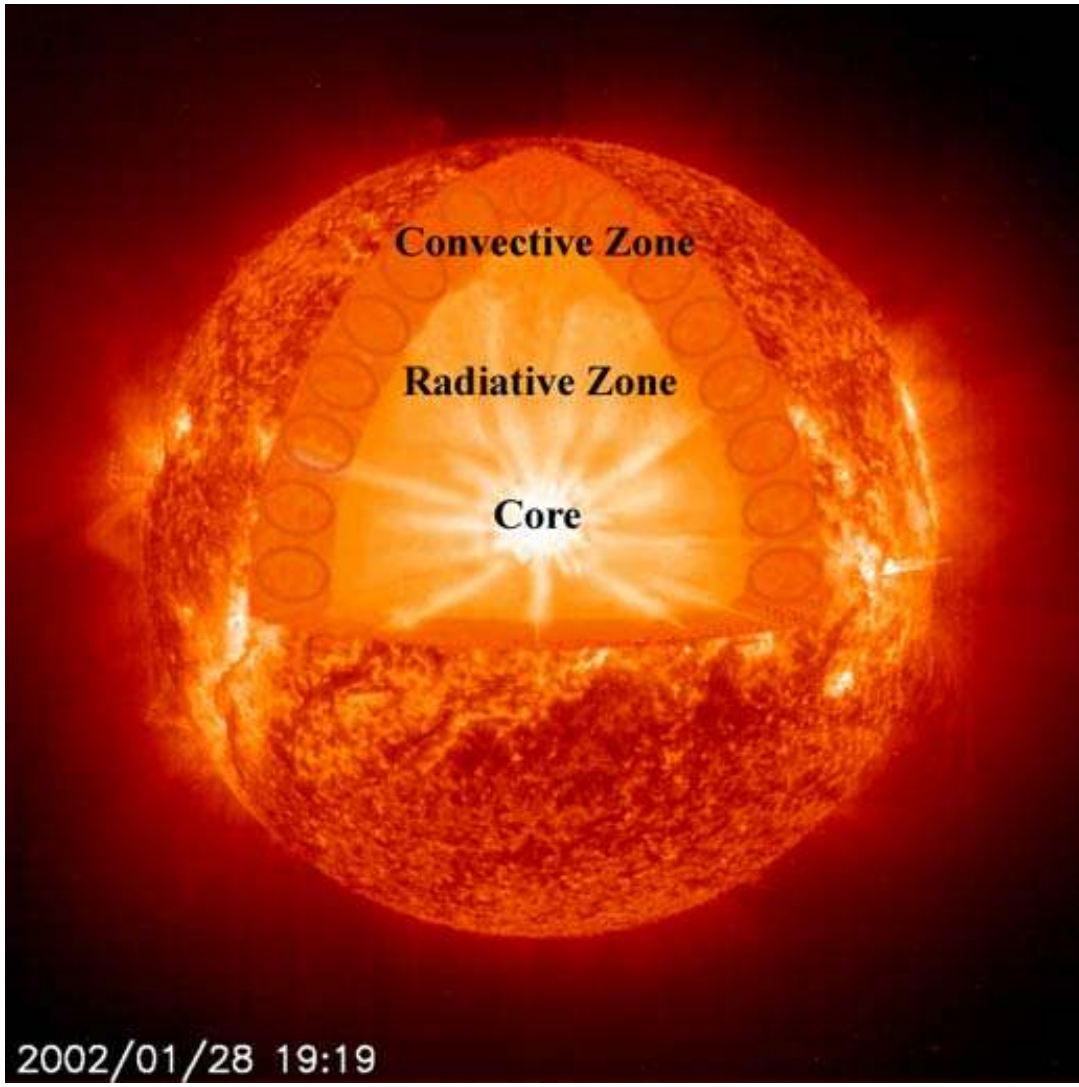
L3: Angular Momentum Transport by IGW

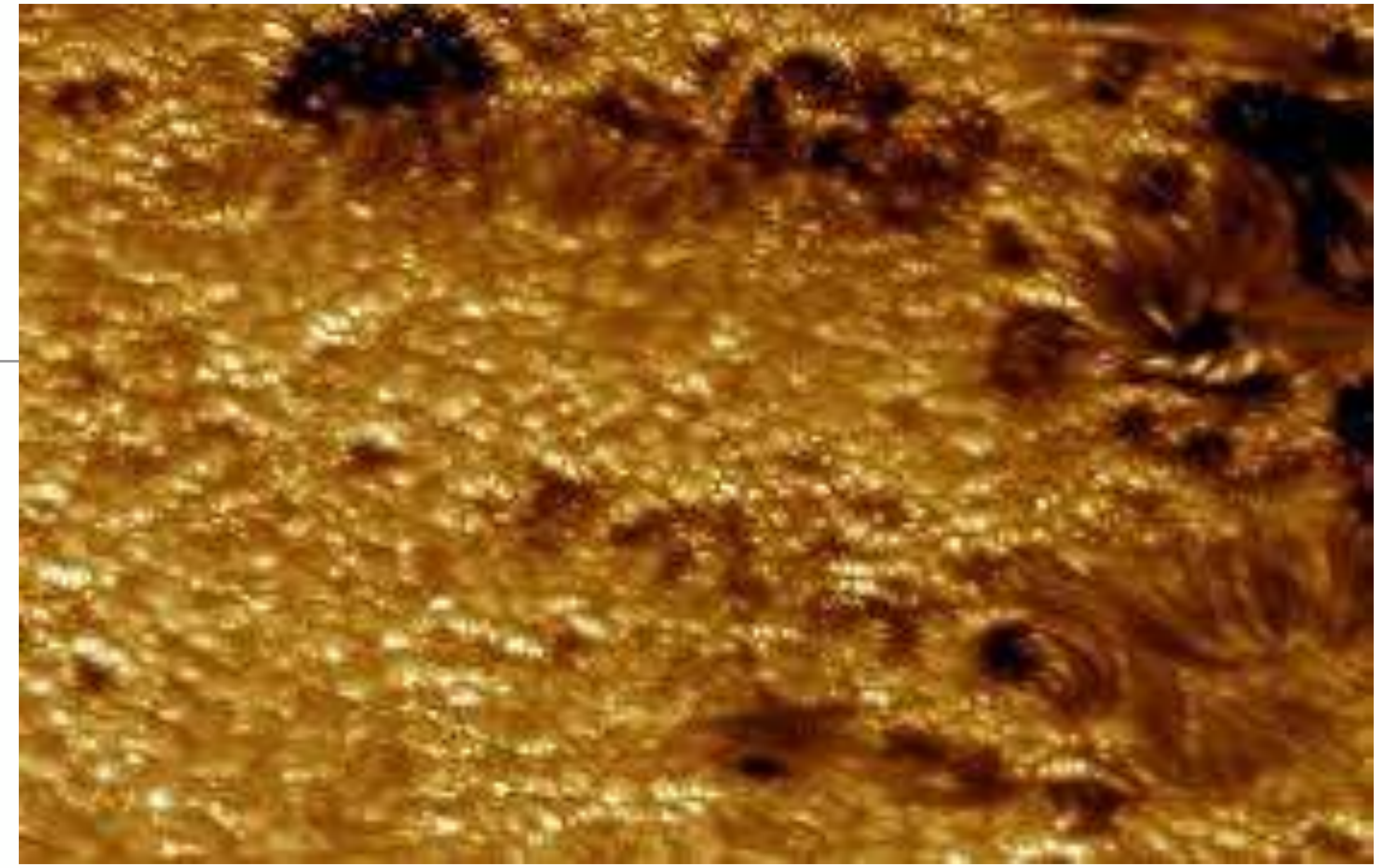
T.M. Rogers

Waves, Instabilities and Turbulence in GAFD, 8-14 July 2019 Cargese

1D Stellar Evolution versus Observation

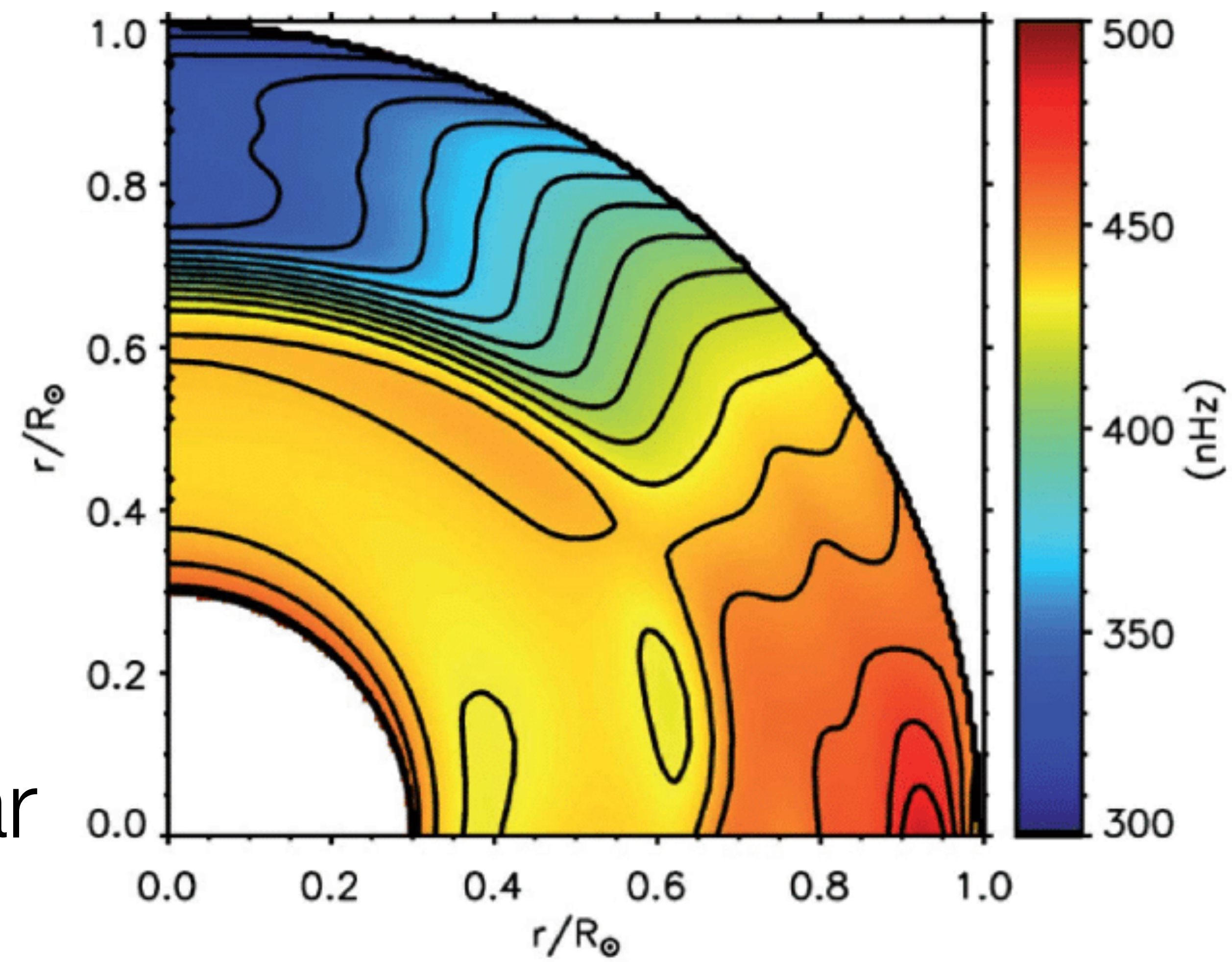






Stars have all manner of (magneto-)hydrodynamics

Most stars are not resolved and what we know is that mixing ***within radiative zones*** is the biggest problem in explaining stellar observations



Angular Momentum Transport by IGW

Start with Momentum Equation

$$\rho\frac{\partial \mathbf{v}}{dt}+\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}=-\nabla-\rho\mathbf{g}\hat{\mathbf{r}}+\rho\nu\left(\nabla^{\mathbf{2}}\mathbf{v}+\mathbf{1/3}\nabla\left(\nabla\cdot\mathbf{v}\right)\right)$$

Take zonal (longitudinal) average in 2D (x,z)

$$u(x,z,t)=\overline{U}+u'(x,z,t)$$

$$w(x,z,t)=w'(x,z,t)$$

Take zonal (longitudinal) average

$$\frac{\partial\left(\rho\overline{U}\right)}{\partial t}=\frac{\partial\left(\rho\overline{u'w'}\right)}{\partial z}+\nu\frac{\partial^2\overline{U}}{\partial z^2}$$

Angular Momentum Transport by IGW

$$\frac{\partial\left(\rho\overline{U}\right)}{\partial t} = \frac{\partial\left(\rho\overline{u'w'}\right)}{\partial z} + \nu\frac{\partial^2\overline{U}}{\partial z^2}$$

Transport AM by the (zonally aveaged)
Reynolds stress

Often called Eliassen-Palm Flux (1960)

$$\frac{\partial\left(\rho\overline{U}\right)}{\partial t} = \frac{\partial F_L}{\partial z} + \nu\frac{\partial^2\overline{U}}{\partial z^2}$$

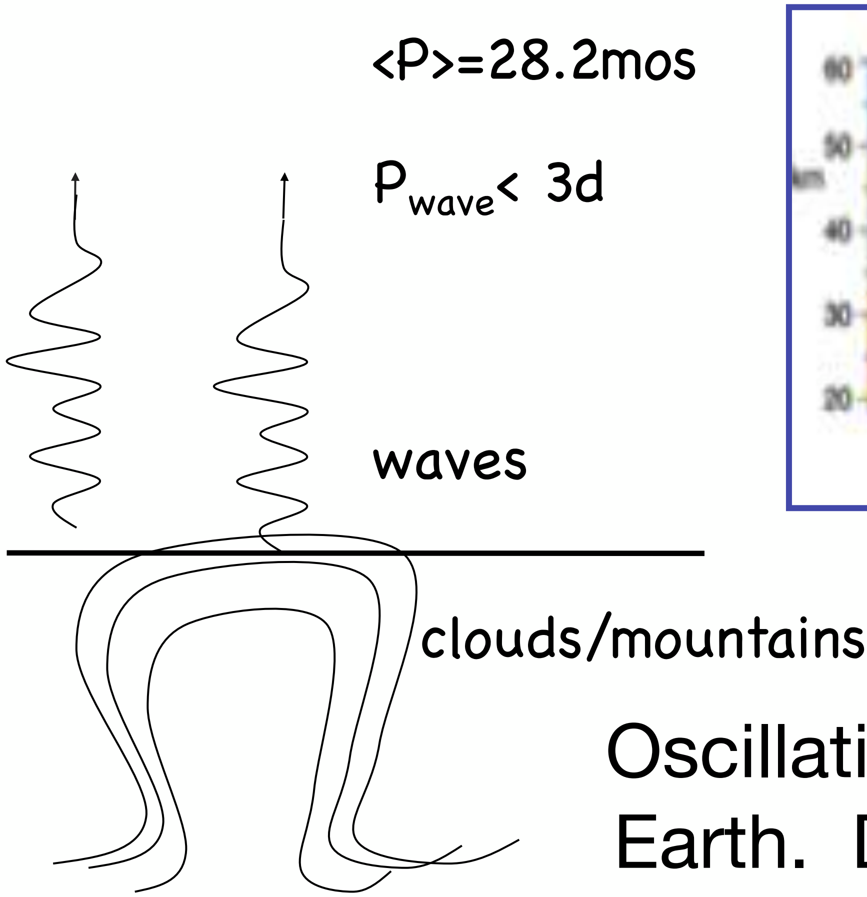
Consider 2 waves, one with + wavenumber m, one with -m

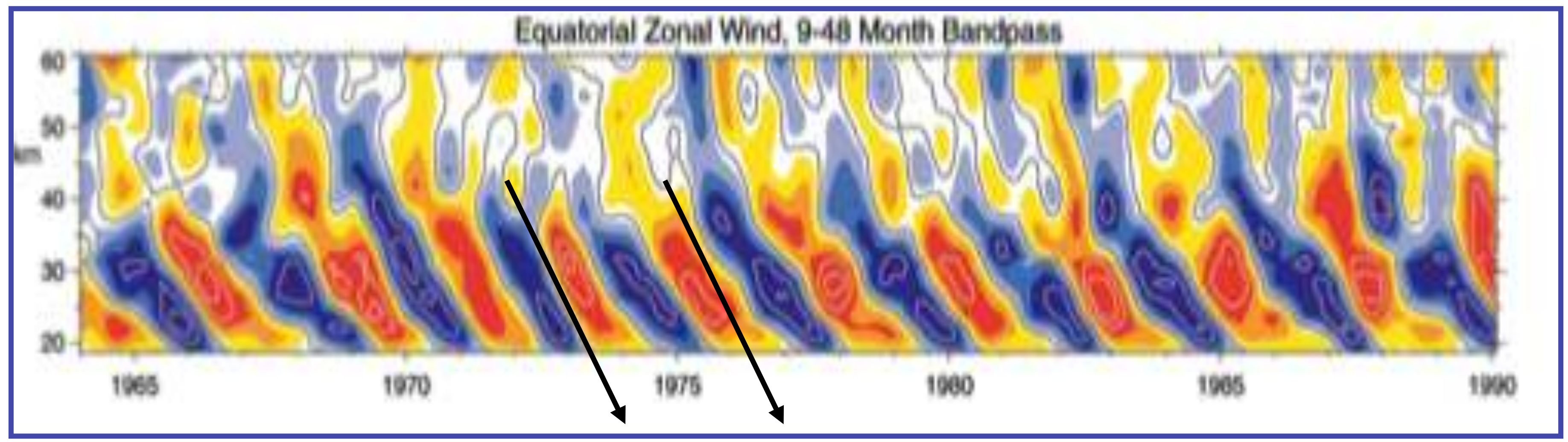
$$u' = A\cos\left(2\pi m x\right) \qquad w' = B\cos\left(2\pi\left(-m\right)x\right)$$

$$u'w' = AB\cos\left(2\pi m x\right)\cos\left(2\pi\left(-m\right)x\right)$$

$$= AB\frac{\cos\left(0\right) + \cos\left(4\pi m x\right)}{2}$$

Quasi Biennial Oscillation



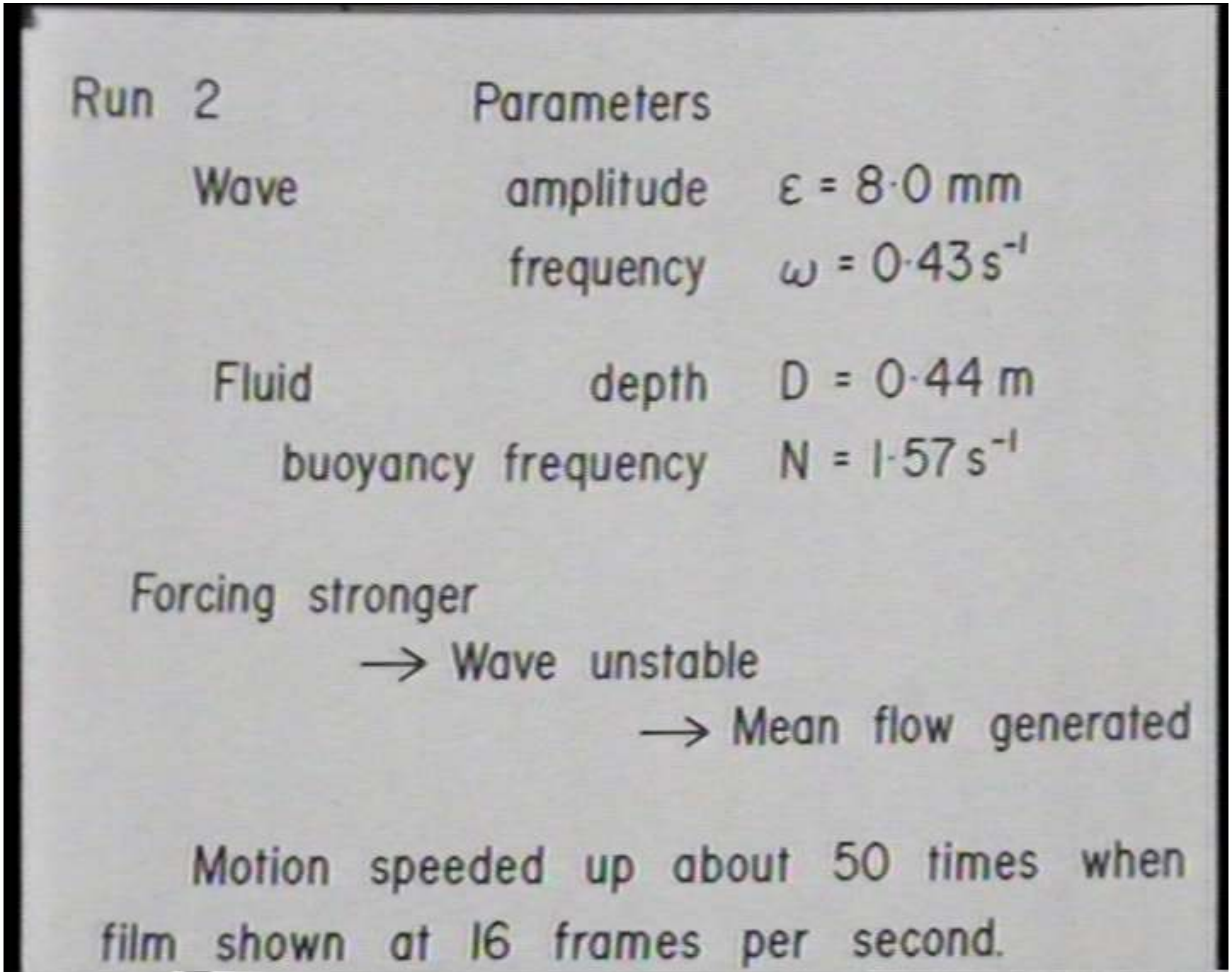


*MOTION TOWARD SOURCE

Baldwin 2001

Oscillation of mean zonal wind in equatorial stratosphere of Earth. Driven by the nonlinear interaction of low-amplitude vertically propagating gravity waves interacting with a background flow

Plumb-McEwan Experiment 1978



Run 2 Parameters

Wave amplitude $\varepsilon = 8{\cdot}0$ mm

frequency $\omega = 0{\cdot}43\,s^{-1}$

Fluid depth $D = 0{\cdot}44$ m

buoyancy frequency $N = 1{\cdot}57\,s^{-1}$

Forcing stronger

→ Wave unstable

→ Mean flow generated

Motion speeded up about 50 times when film shown at 16 frames per second.

Bottom boundary driven with ± 4

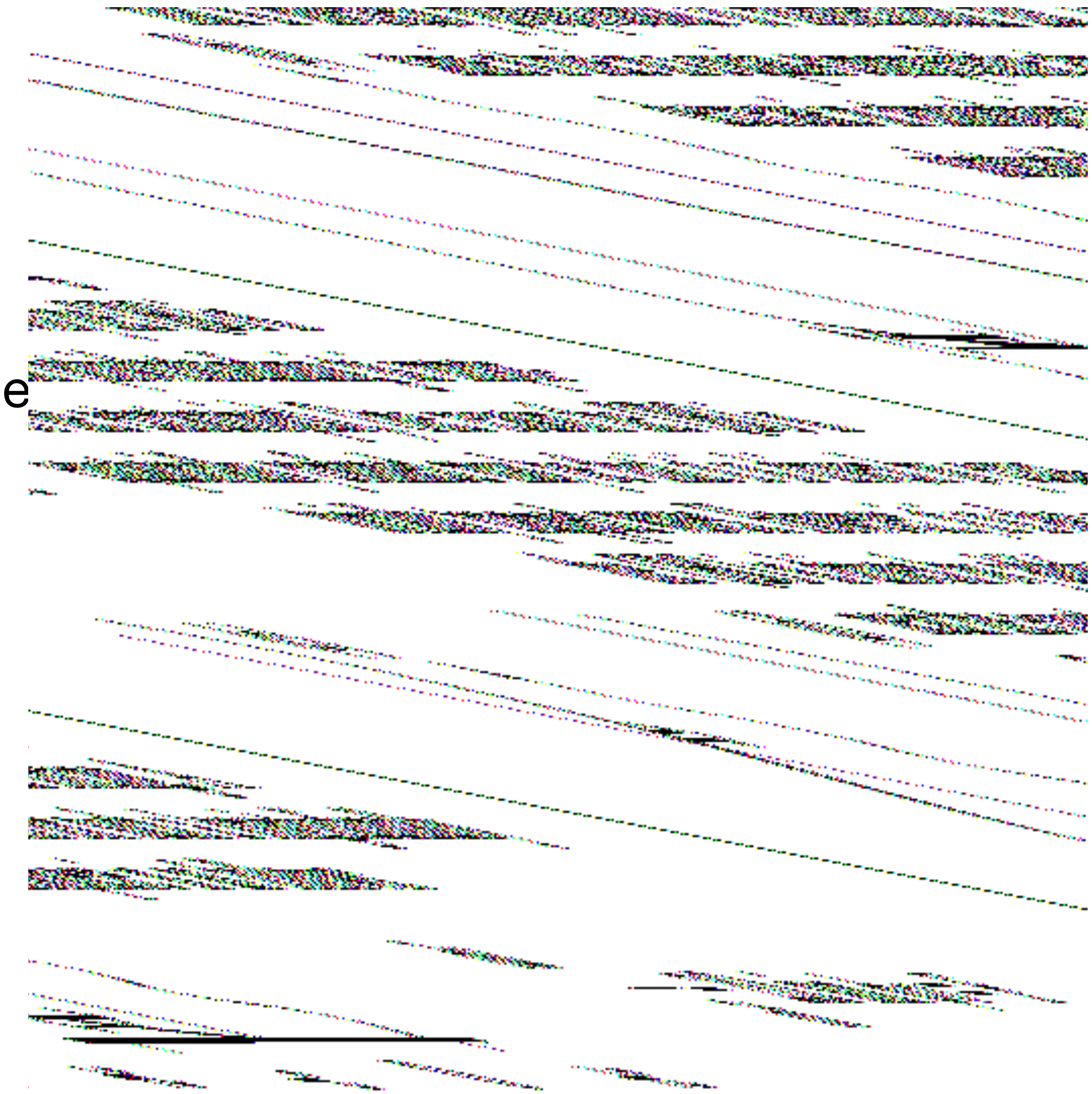
IGW drive mean shear flows which oscillate in time and migrate toward the source of the waves.

Waves in Differential Rotation

- Waves are Doppler shifted in a differentially rotating medium

$$\omega\left(r\right)=\omega_{gen}+m\left[\Omega_{gen}-\Omega\left(r\right)\right]$$

- When rotation increasing, a prograde (+m) wave will be shifted to lower frequencies (damped earlier). Conversely, a retrograde wave (-m), shifted to higher frequency, propagates further (a)
- This enhances differential rotation until viscosity destroys shear at bottom (b)
- Process repeats (reversed)



Baldwin et al. 2001
adapted from Plumb

What we learn from QBO/Plumb&McEwan

- Waves transport AM from where they are generated, to where they are dissipated
- Waves drive shear flows

In stars this means waves can efficiently couple (in the AM sense) convective and radiative regions

BUT Angular Momentum Transport by IGW cannot be treated as a diffusion coefficient (but maybe for mixing, see next lecture)

Waves in Stars (spheres)

(Zahn et al. 1997)

Zonal Average of Momentum Equation

$$\mathbf{U}(r,\theta,\phi,t) = \mathbf{\Omega}(r)\,\hat{\mathbf{z}} \times \mathbf{r} + \mathbf{u}(r,\theta,\phi,t)$$

Mean Flow Wave

$$\rho \frac{d\left(r^2\overline{\Omega}\right)}{dt} = -\frac{1}{r^2}\frac{\partial\left(r^2\overline{\rho r \sin\theta v_r v_\phi}\right)}{\partial r} + \frac{1}{r^4}\frac{\partial}{\partial r}\left(\rho\nu r^4 \frac{\partial\overline{\Omega}}{\partial r}\right)$$

$$\rho \frac{d\left(r^2\overline{\Omega}\right)}{dt} = \frac{dF_L}{dr} + \frac{1}{r^4}\frac{\partial}{\partial r}\left(\rho\nu r^4 \frac{\partial\overline{\Omega}}{\partial r}\right)$$

$$F_{K\omega} = KE \times v_g$$

$$F_{L\omega} = 2\frac{m}{\omega}F_K$$

$$F_L = \sum_{l,m,\omega} F_{L\omega}$$

Waves in Stars (spheres)

(Zahn et al. 1997)

$$\rho \frac{d\left(r^2\overline{\Omega}\right)}{dt} = \frac{dF_L}{dr} + \frac{1}{r^4}\frac{\partial}{\partial r}\left(\rho \nu r^4 \frac{\partial \overline{\Omega}}{\partial r}\right)$$

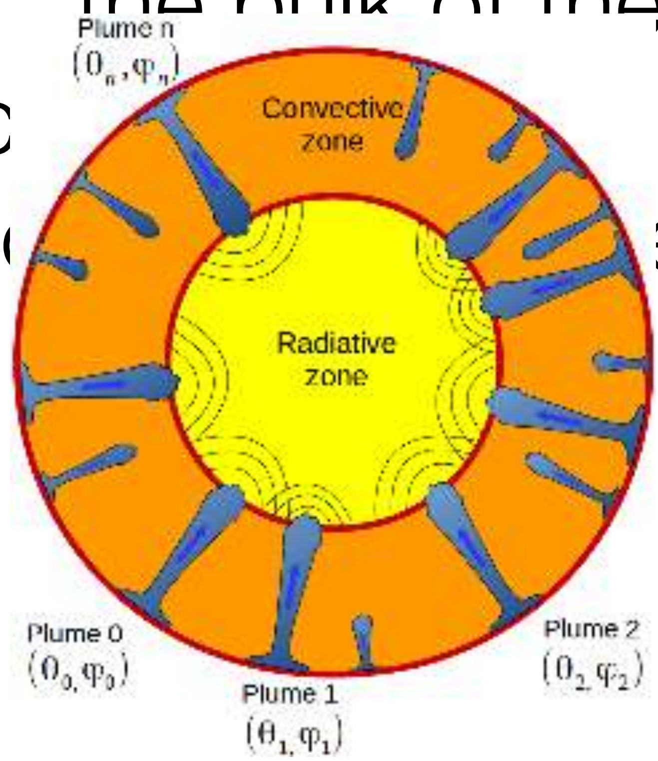
How can we estimate this flux in a star?

1. We need to know amplitudes of waves at ***generation*** (the spectrum)
2. We need to know how these amplitudes vary with radius ***(propagation)***
3. We need to know how the waves ***dissipate***

Wave Generation in Stars

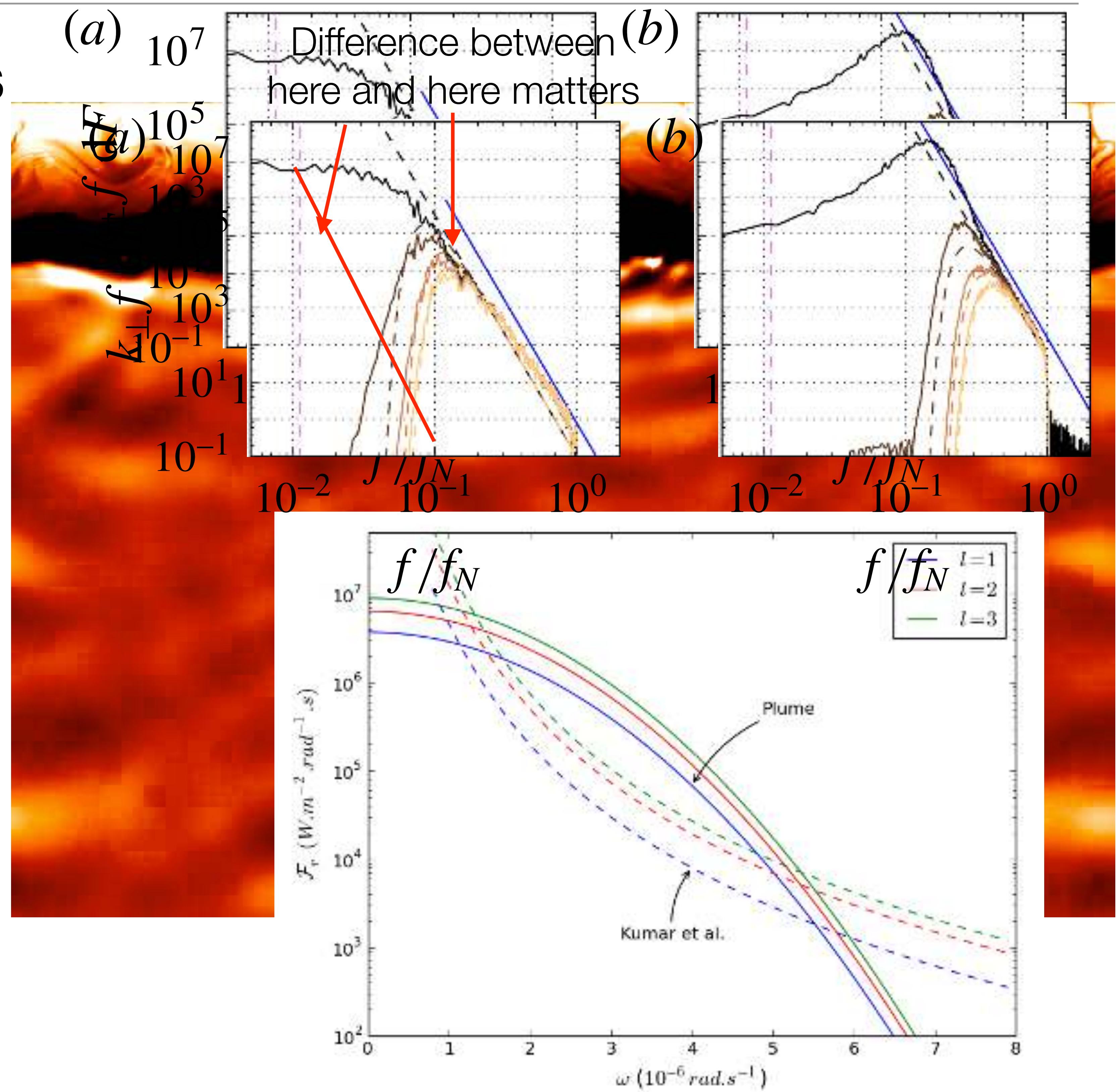
Daniel's talk:
There are other ideas

waves generated in
the bulk of the
c by
es



Plumes/Overshoot

Pincon et al. 2016



Constructing Wave Flux

$$u_0(\omega, \ell) \propto \omega^m \left(\sqrt{\ell(\ell+1)}\right)^n$$

Spectra	m	n
Flat[F]	0	0
Kumar et al. (1999)[K]	-2.17	1
Lecoanet & Quataert (2013)[LD]	-4.25	2.5
Rogers et al. (2013)[R]	-0.6	-0.9

Wave Propagation in Stars
(Zahn et al. 1997)

Go back to Momentum, Energy, Continuity Equations (in spherical coordinates,anelastic approximation), linearize

$$\frac{d^2\Psi}{dr^2} + \left(\frac{N^2}{\omega^2} - 1\right)\frac{\ell\left(\ell+1\right)}{r^2}\Psi = 0$$

$$\Psi = \rho^{1/2} r^2 u_v$$

Solve WKB (low frequency, small vertical wavelength) with dissipation

$$u_r = C r^{-\frac{3}{2}} \boxed{\rho^{-\frac{1}{2}}} \left(\frac{N^2}{\omega^2} - 1\right)^{-\frac{1}{4}} \boxed{e^{-\tau}} P_l^m\left(\cos\theta\right) \times \cos\left(\omega t - m\left(\phi - \Omega t\right) - \int_{r1}^{r2} k_v \, dr\right)$$

Density Stratification Important

Radiative Dissipation

Wave Frequencies are Doppler Shifted in Differential Rotation

$$\omega\left(r\right) = \omega_{gen} + m\left[\Omega_{gen} - \Omega\left(r\right)\right]$$

IGW Dissipation in Stars (Thermal Diffusion)

(Zahn et al. 1997)

$$u_r = C r^{-\frac{3}{2}} \rho^{-\frac{1}{2}} \left(\frac{N^2}{\omega^2} - 1 \right)^{-\frac{1}{4}} e^{-\tau} P_l^m (\cos\theta) \times \cos\left(\omega t - m(\phi - \Omega t) - \int_{r1}^{r2} k_v \, dr \right)$$

Radiative Dissipation

$$\tau(\omega, \ell, r) = \int dr \frac{\gamma}{v_g} = \frac{\text{Dissipation Rate}}{\text{Vertical Group Velocity}} = \int dr\, K \frac{k_h^4 N^3}{\omega^4}$$

Thermal Diffusivity - strong function of radius

$$k_h = \frac{\ell(\ell+1)}{r^2}$$

$$l_d \sim \frac{v_g}{\gamma} = \frac{\omega^4}{k_h^3 N^3 \kappa}$$

Low frequencies/small scales damp rapidly
High frequencies/large scales propagate further

Critical Layers

Critical Layer is defined as the radius where the mean zonal velocity is equal to the horizontal phase speed of the wave

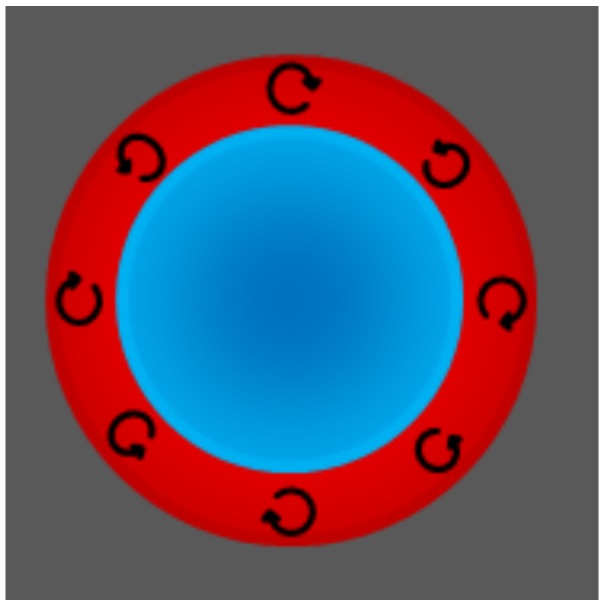
$$\overline{U} = c_{ph}$$

When the Richardson number is large, there is nearly complete wave momentum transferred to the mean flow (Booker & Bretherton (1967) but some energy may be transmitted or reflected (Winters & D'Asaro 1989)

Mean Flow Acceleration at a critical layer is rapid and local

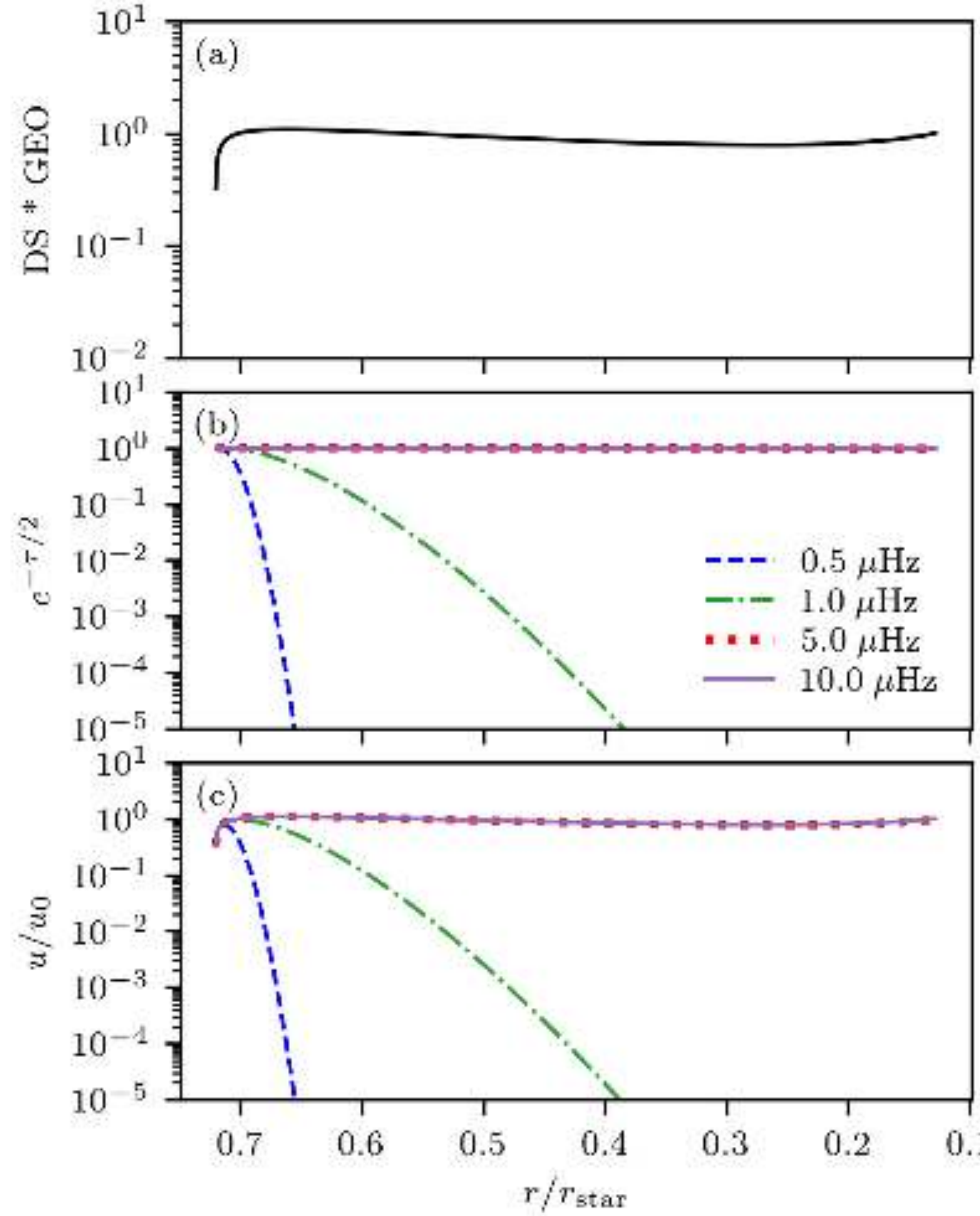
Given the broad spectrum of waves generated by convection it is possible (likely?) that critical layers do form in stars and contribute to angular momentum transport (see next lecture)

So what happens in Stars (Solar)



Density Stratification & Geometry

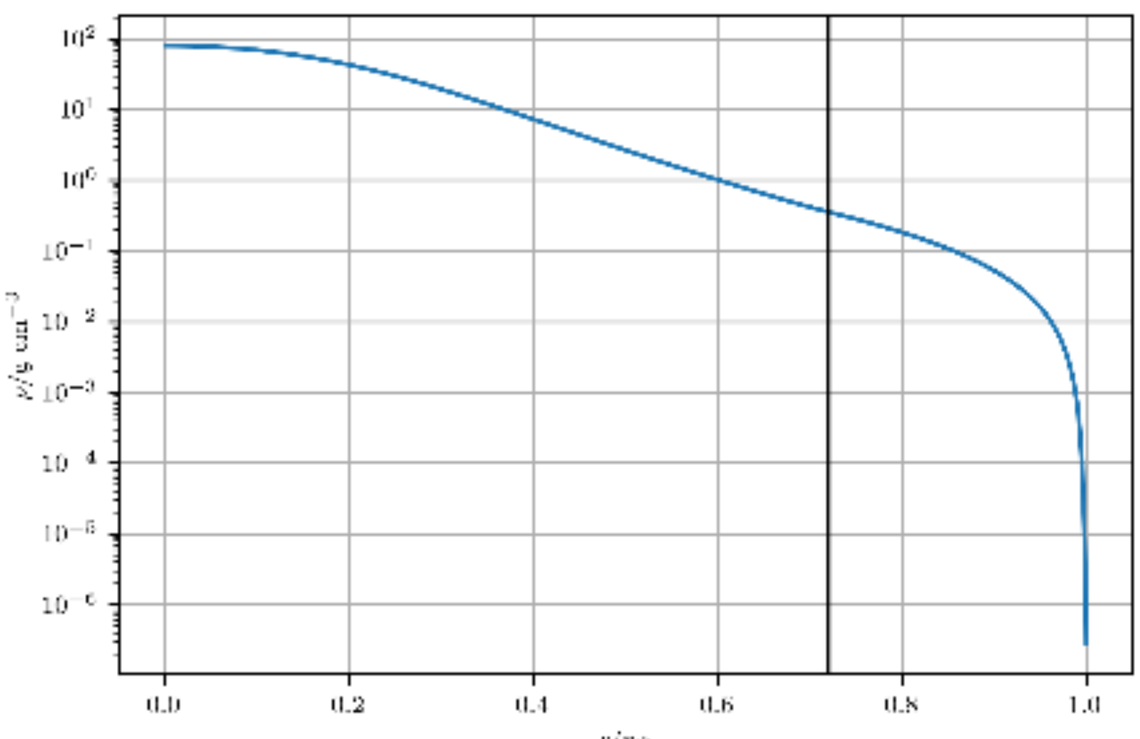
$$\rho^{-1/2} r^{-3/2}$$



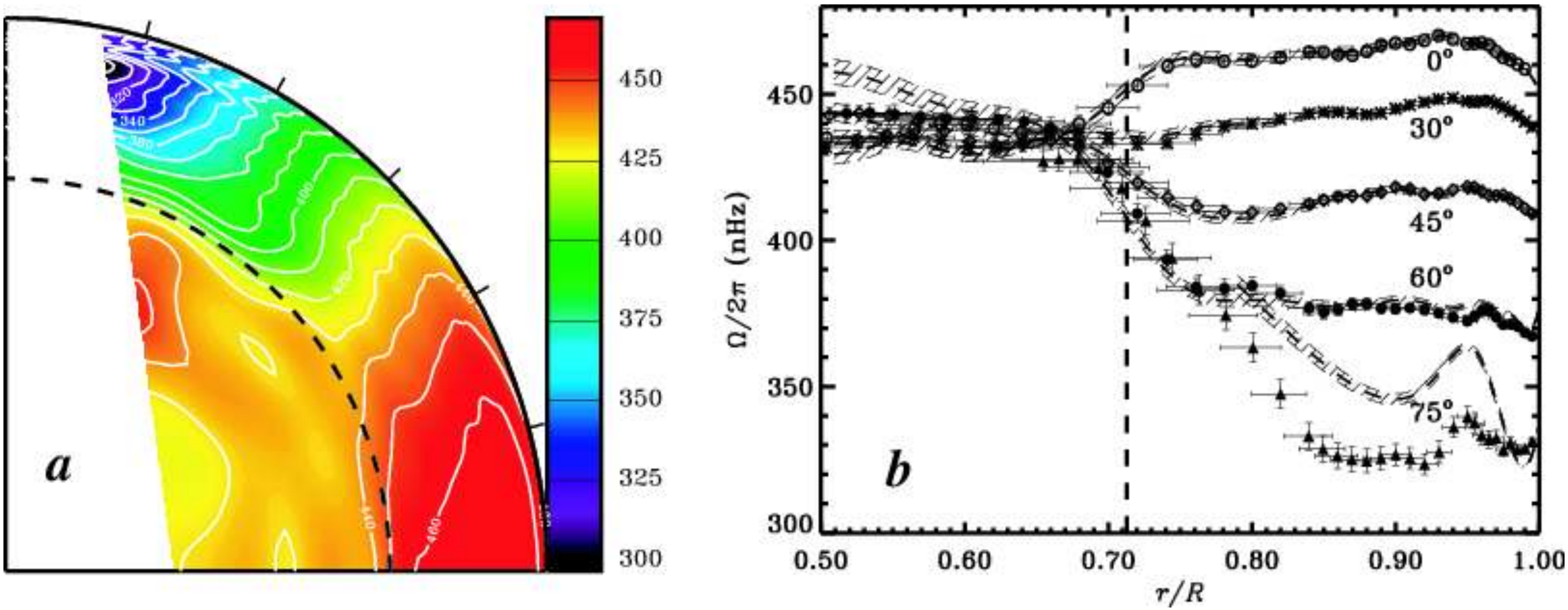
Radiative Diffusion

$$e^{-\tau/2}$$

Combined



How does this work in the Sun?



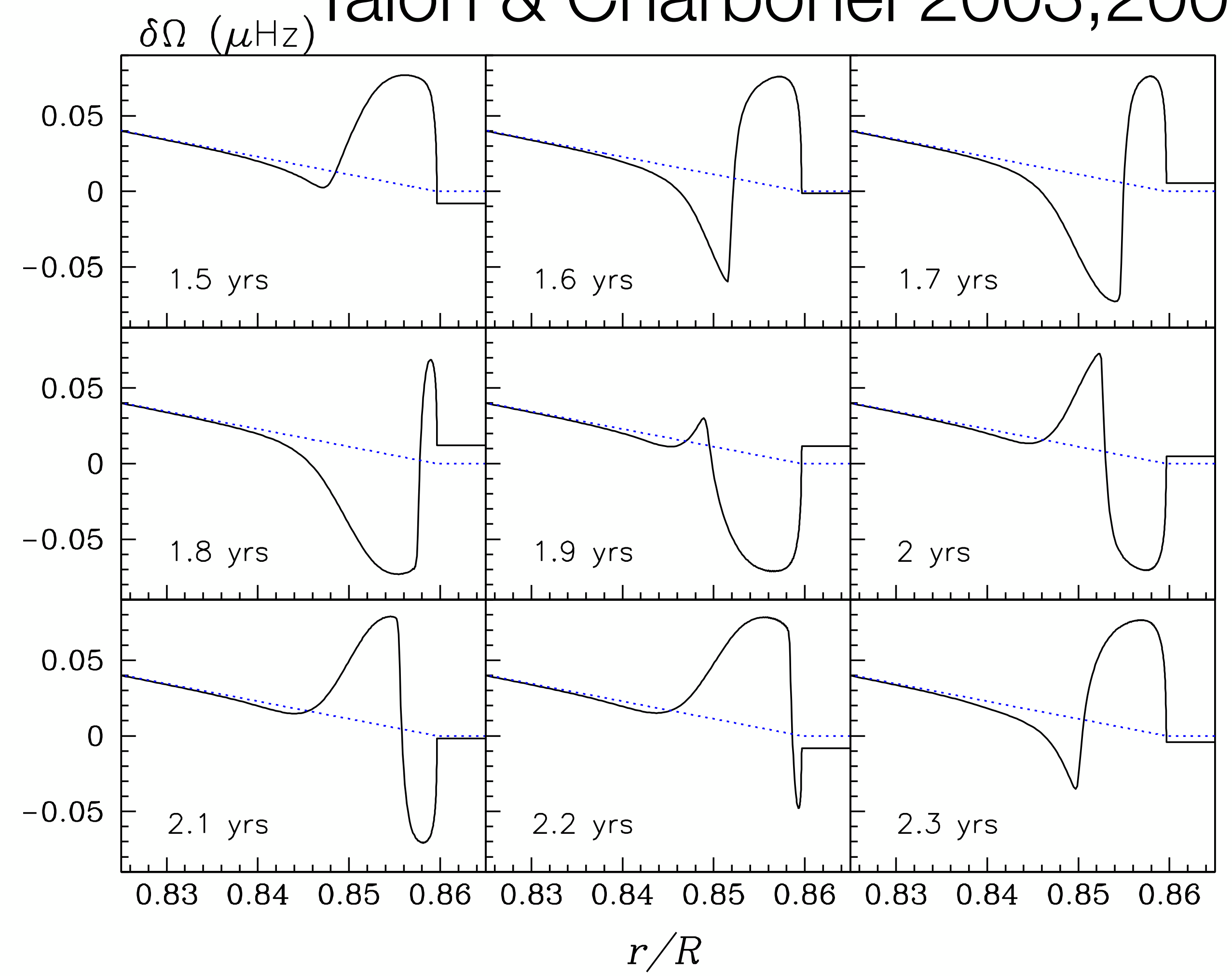
Theoretical Calculations in the Sun

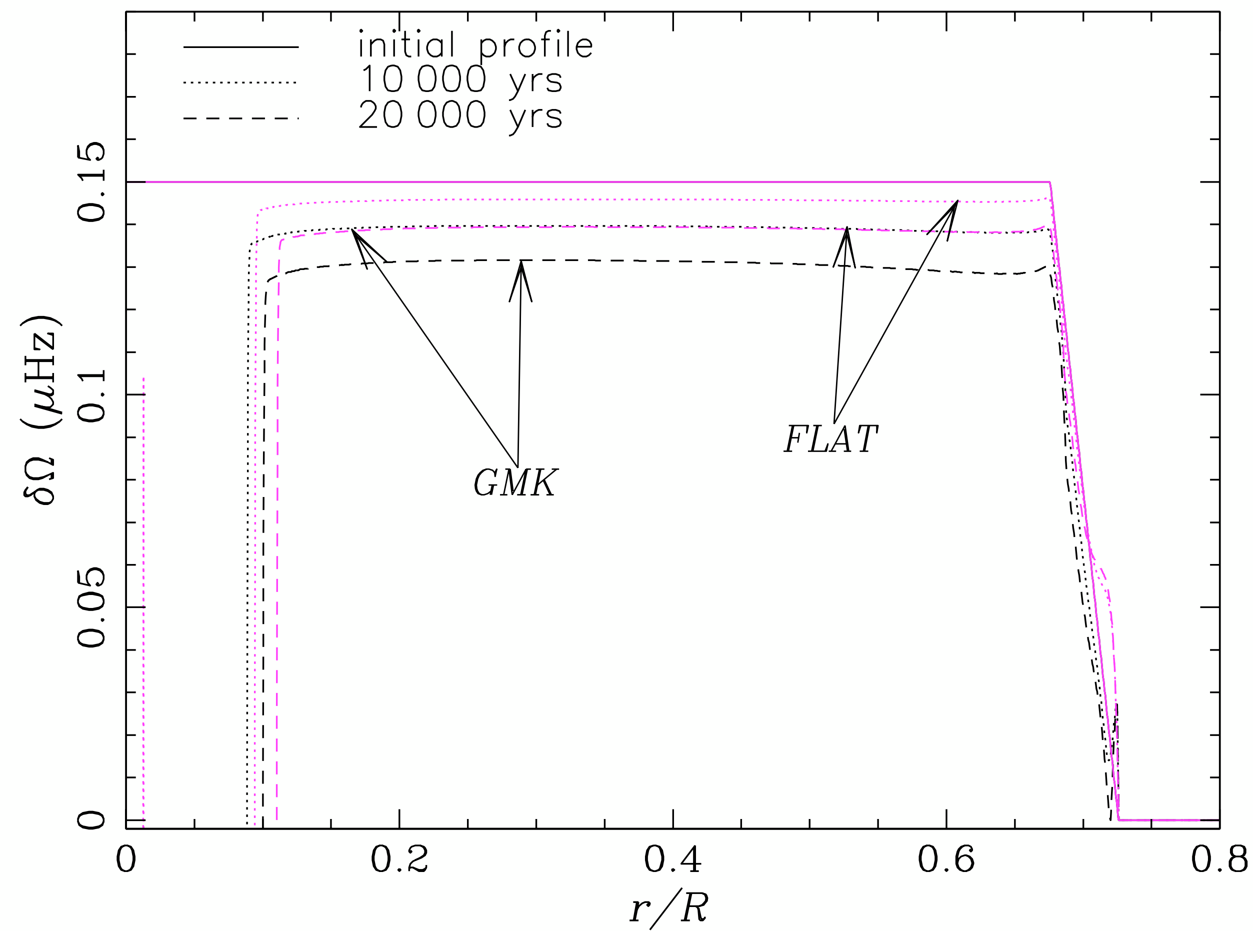
Talon & Charbonel 2003,2005

- Construct Flux from Kumar spectrum and theory described previously

$$\rho \frac{d\left(r^2\overline{\Omega}\right)}{dt} = \frac{dF_L}{dr} + \frac{1}{r^4}\frac{\partial}{\partial r}\left(\rho\nu r^4 \frac{\partial\overline{\Omega}}{\partial r}\right)$$

- Solve on short timescales near the convection zone get SLO (QBO analogue)
- Solve on long timescales in deeper interior - get flat profile in interior
- BUT only works for high viscosity

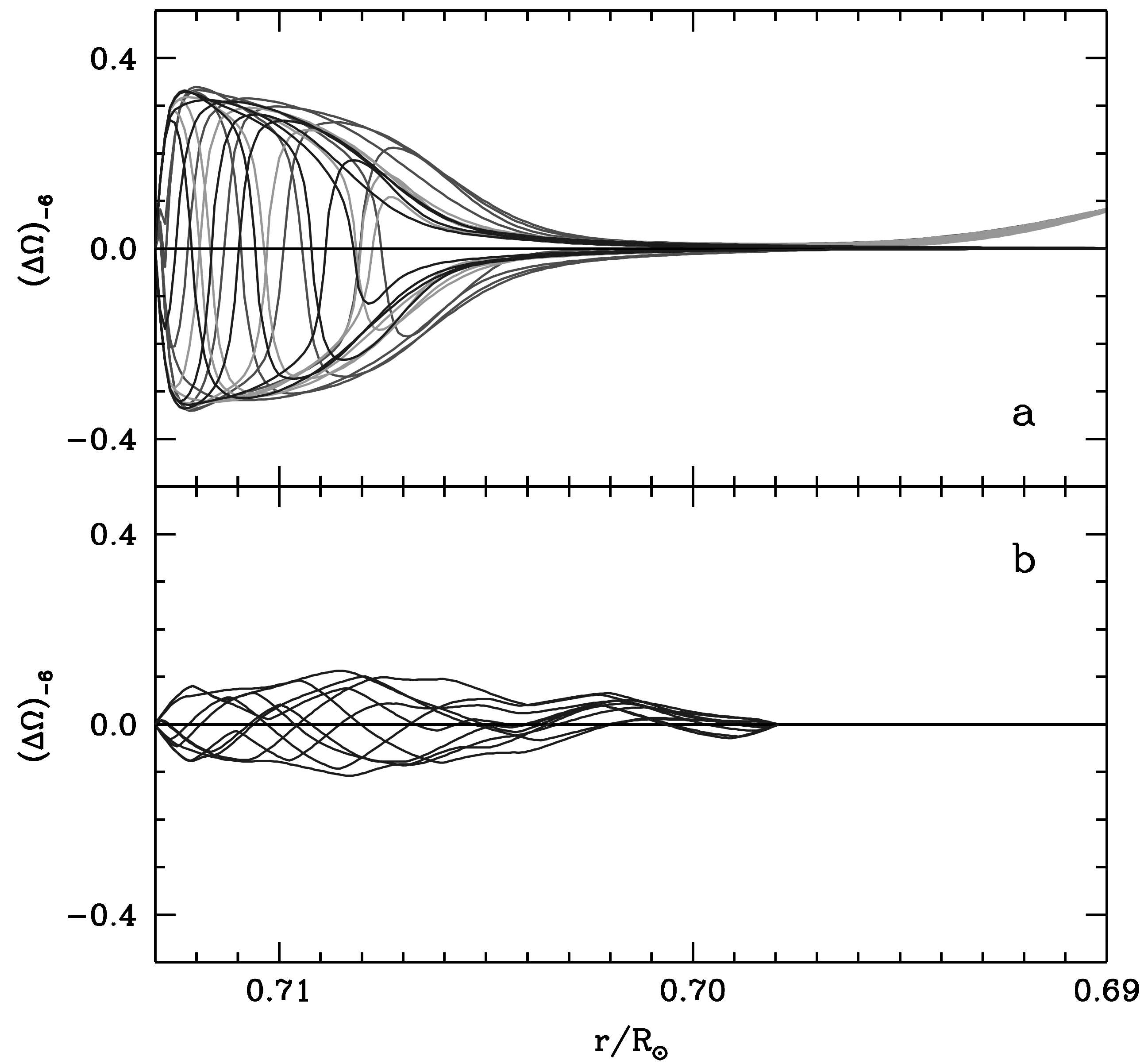


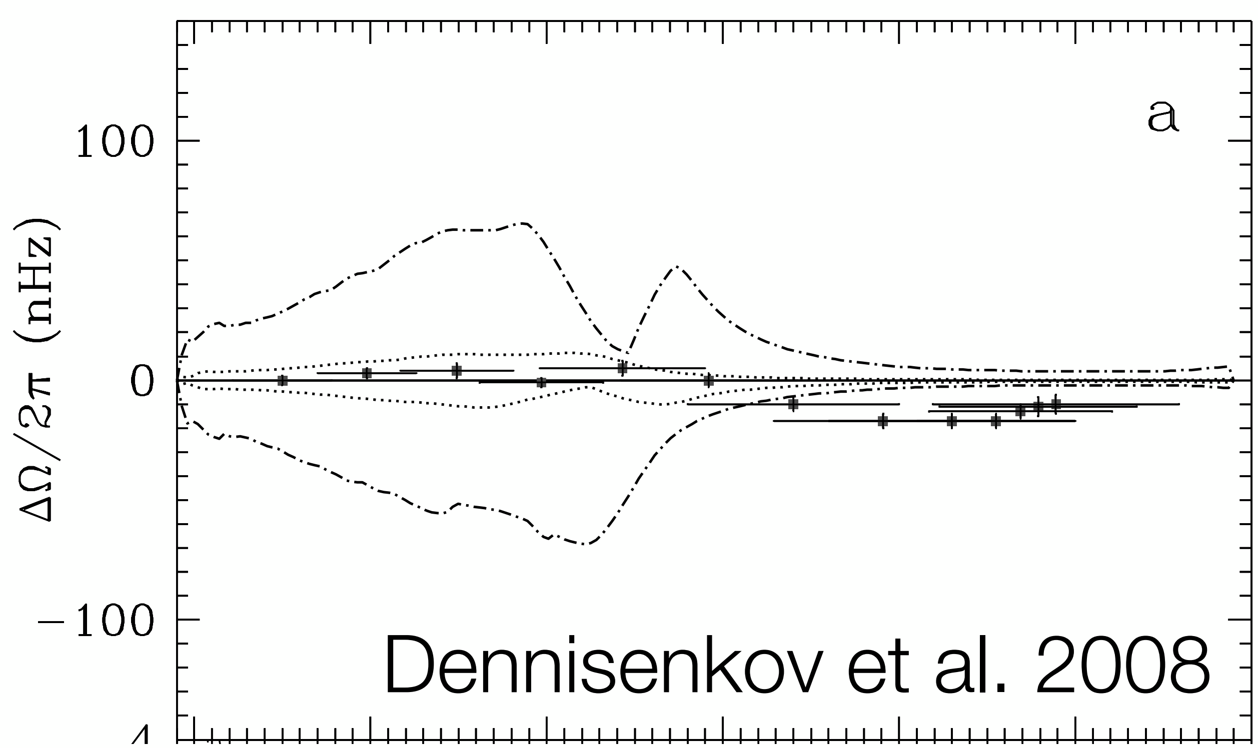


Theoretical Calculations in the Sun

- If use more realistic viscosity, SLO amplitude is decreased AND force differential rotation in Solar interior- counter to observations

- Numerical Simulations of solar interior get no SLO and do not get uniform rotation of solar interior

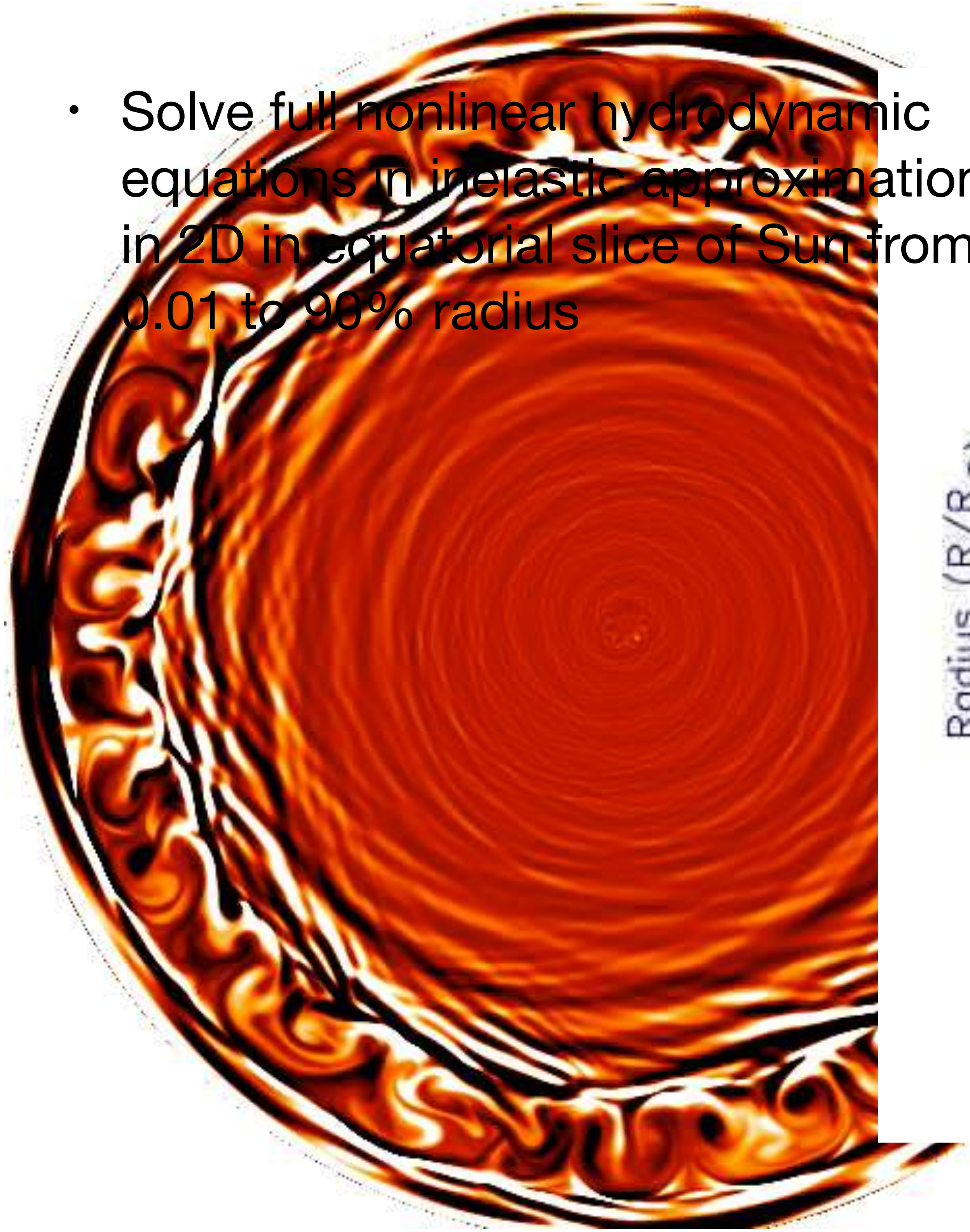


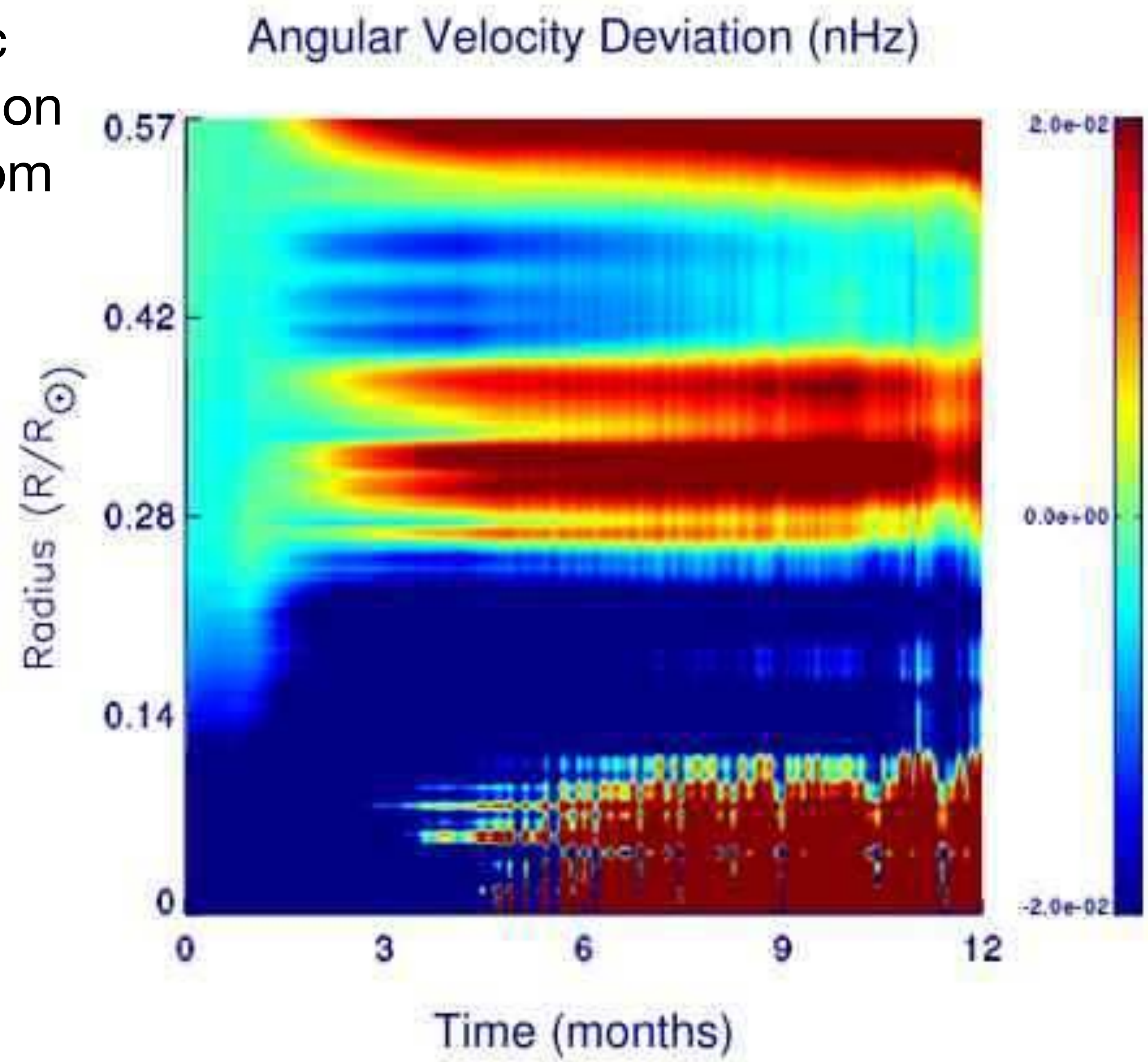


Dennisenkov et al. 2008

Numerical Calculations in the Sun

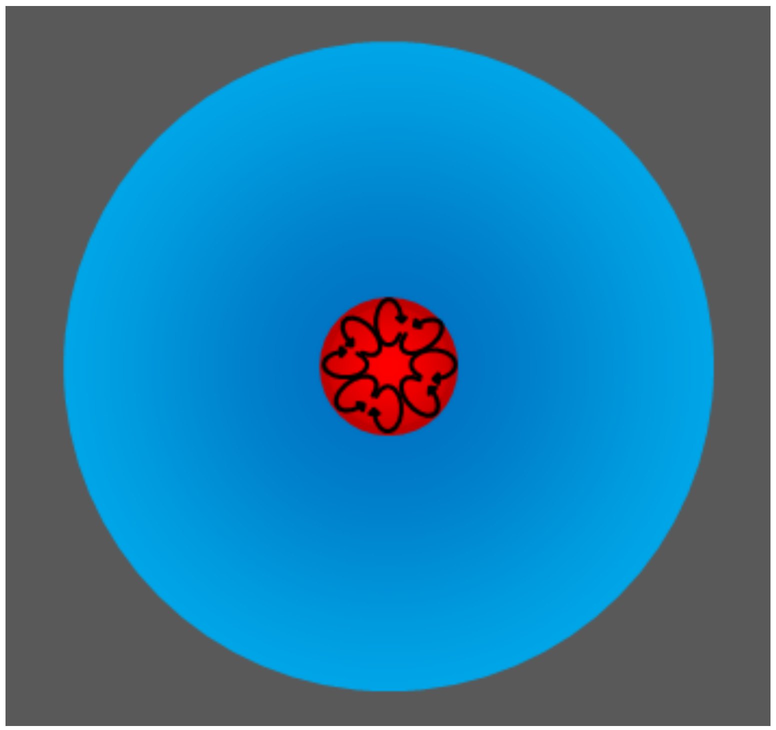
- Solve full nonlinear hydrodynamic equations in inelastic approximation in 2D in equatorial slice of Sun from 0.01 to 90% radius

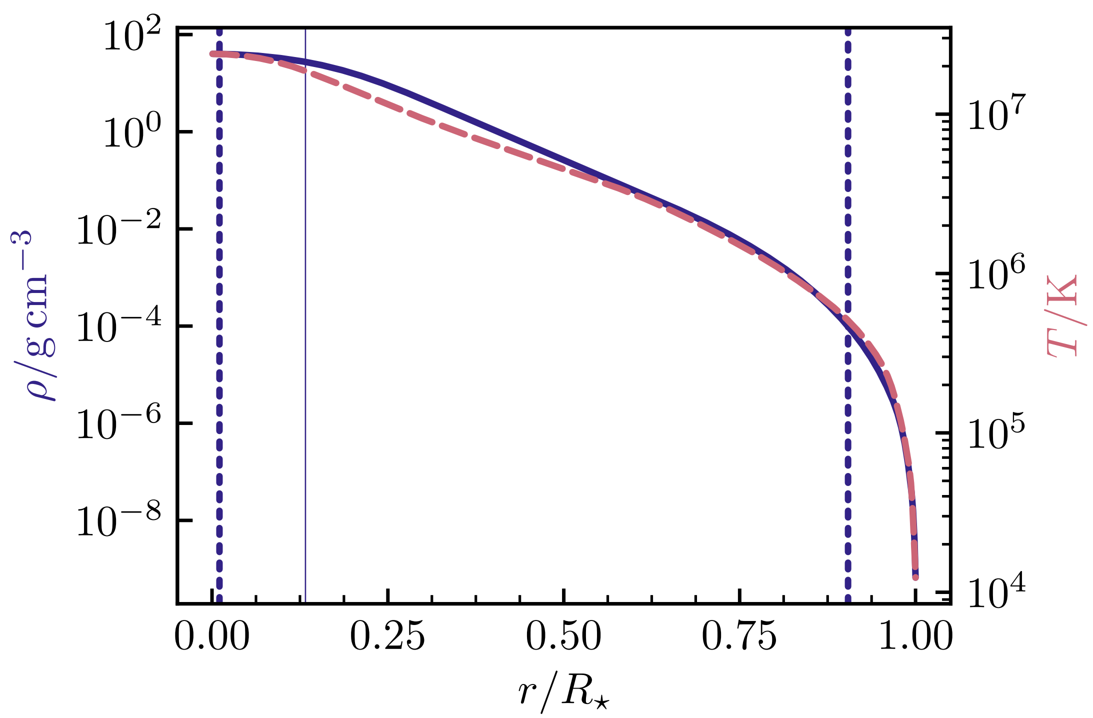




So what happens in Stars (Massive)

$$u_r = C r^{-\frac{3}{2}} \rho^{-\frac{1}{2}} \left(\frac{N^2}{\omega^2} - 1 \right)^{-\frac{1}{4}} e^{-\tau} P_l^m (\cos\theta) \times \cos\left(\omega t - m(\phi - \Omega t) - \int_{r1}^{r2} k_v \, dr \right)$$





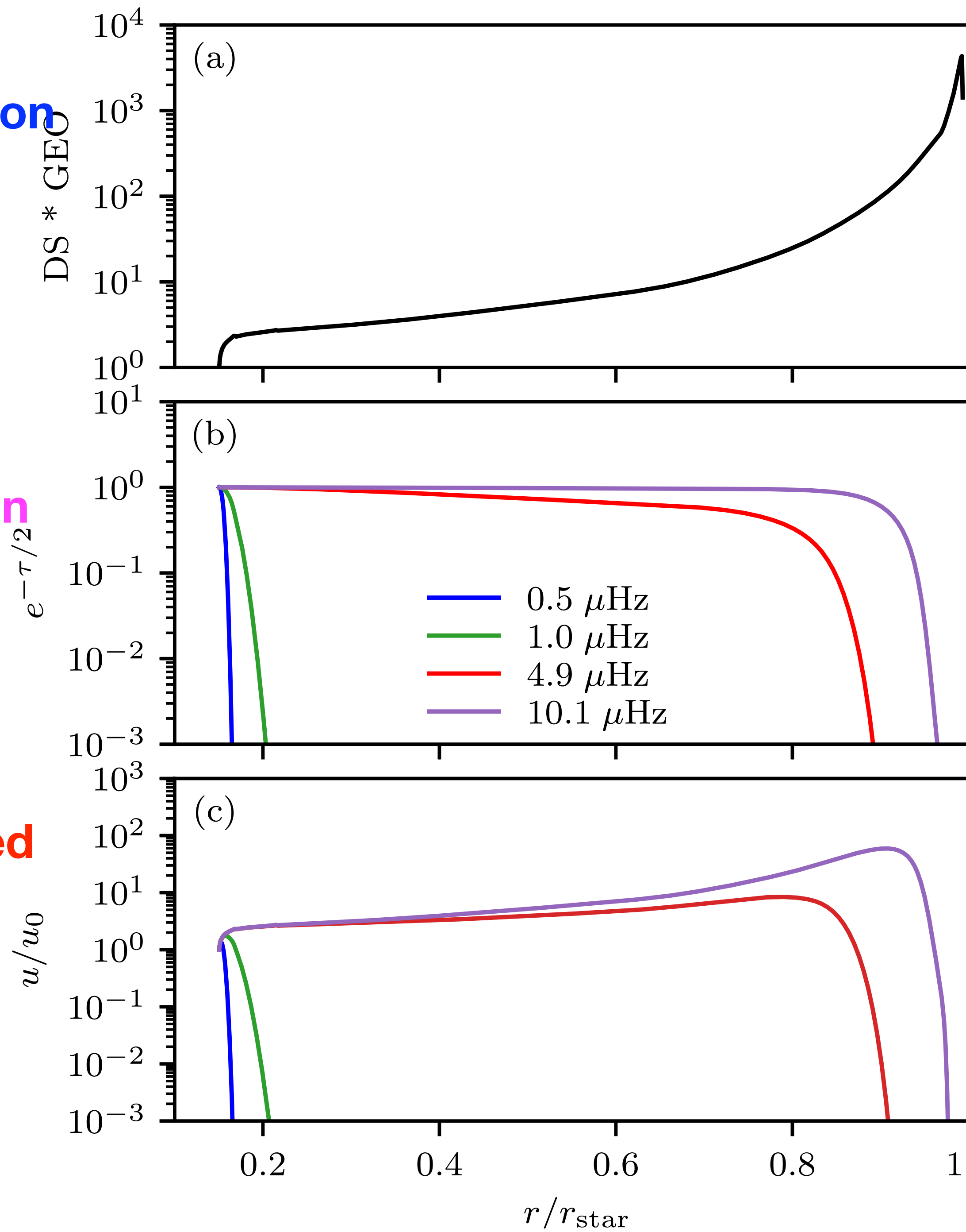
Density Stratification & Geometry

$\rho^{-1/2} r^{-3/2}$

Radiative Diffusion

$e^{-\tau/2}$

Combined

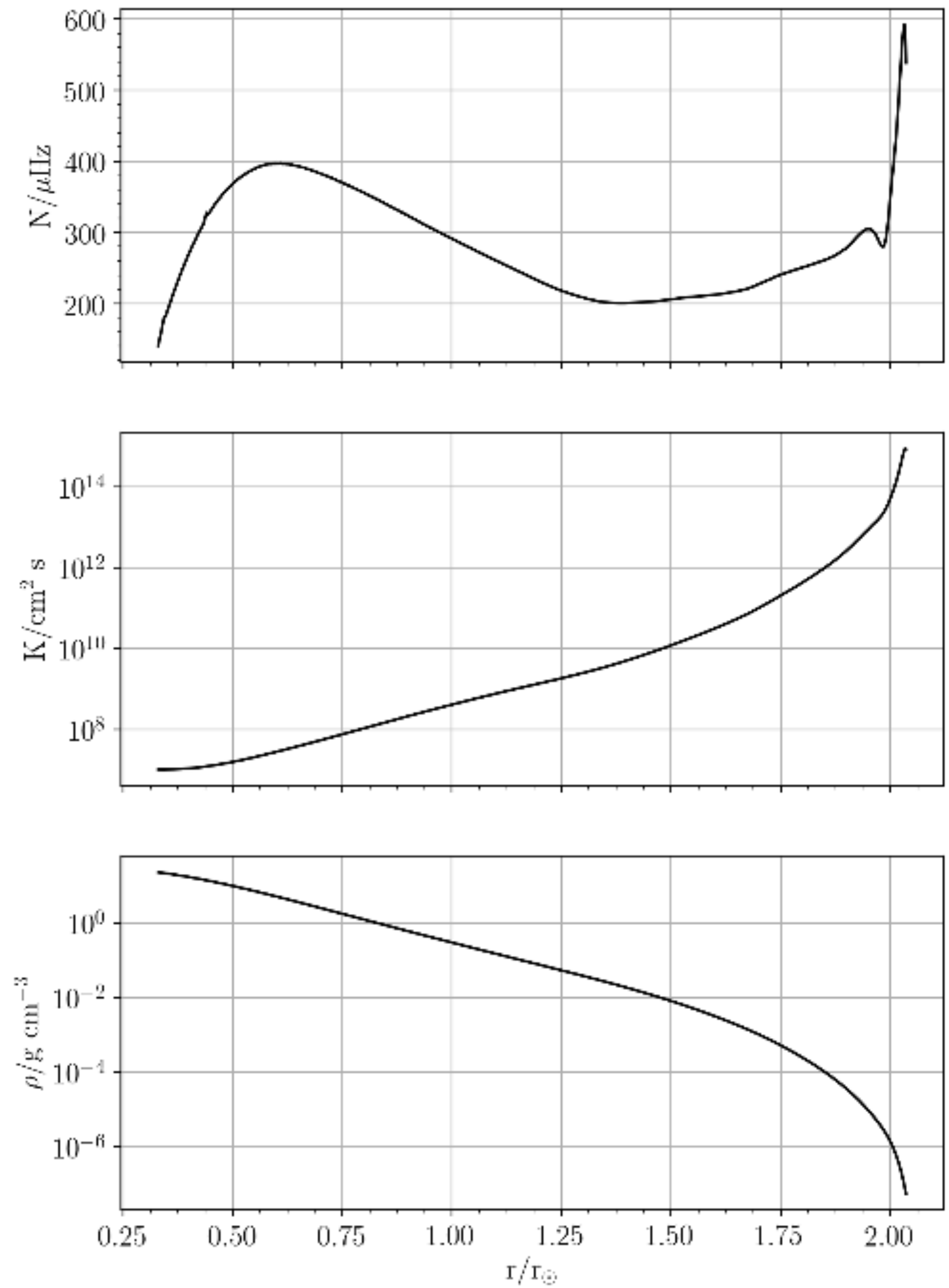


Ratnasingham et al. 2018

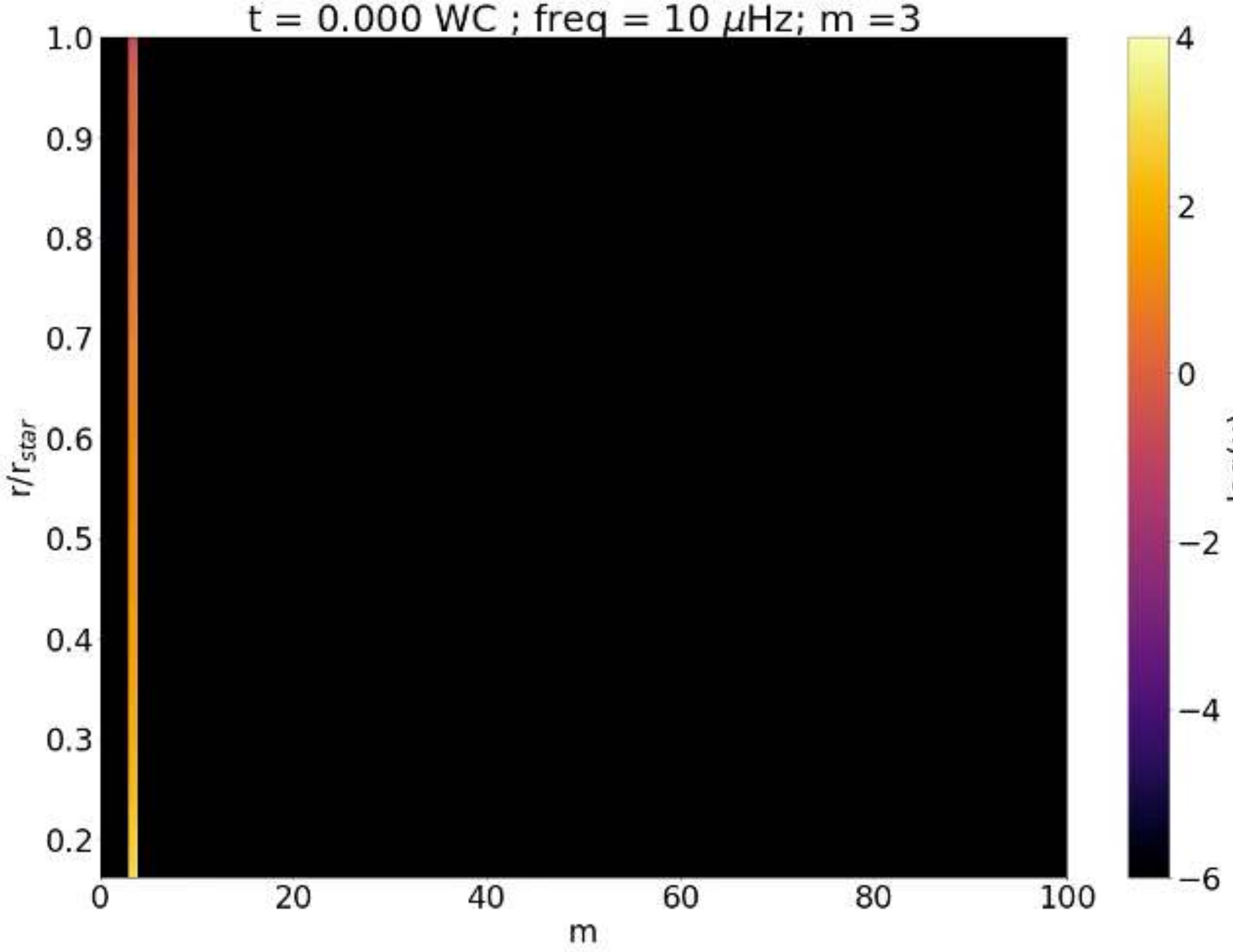
Forcing a single wave in a massive star

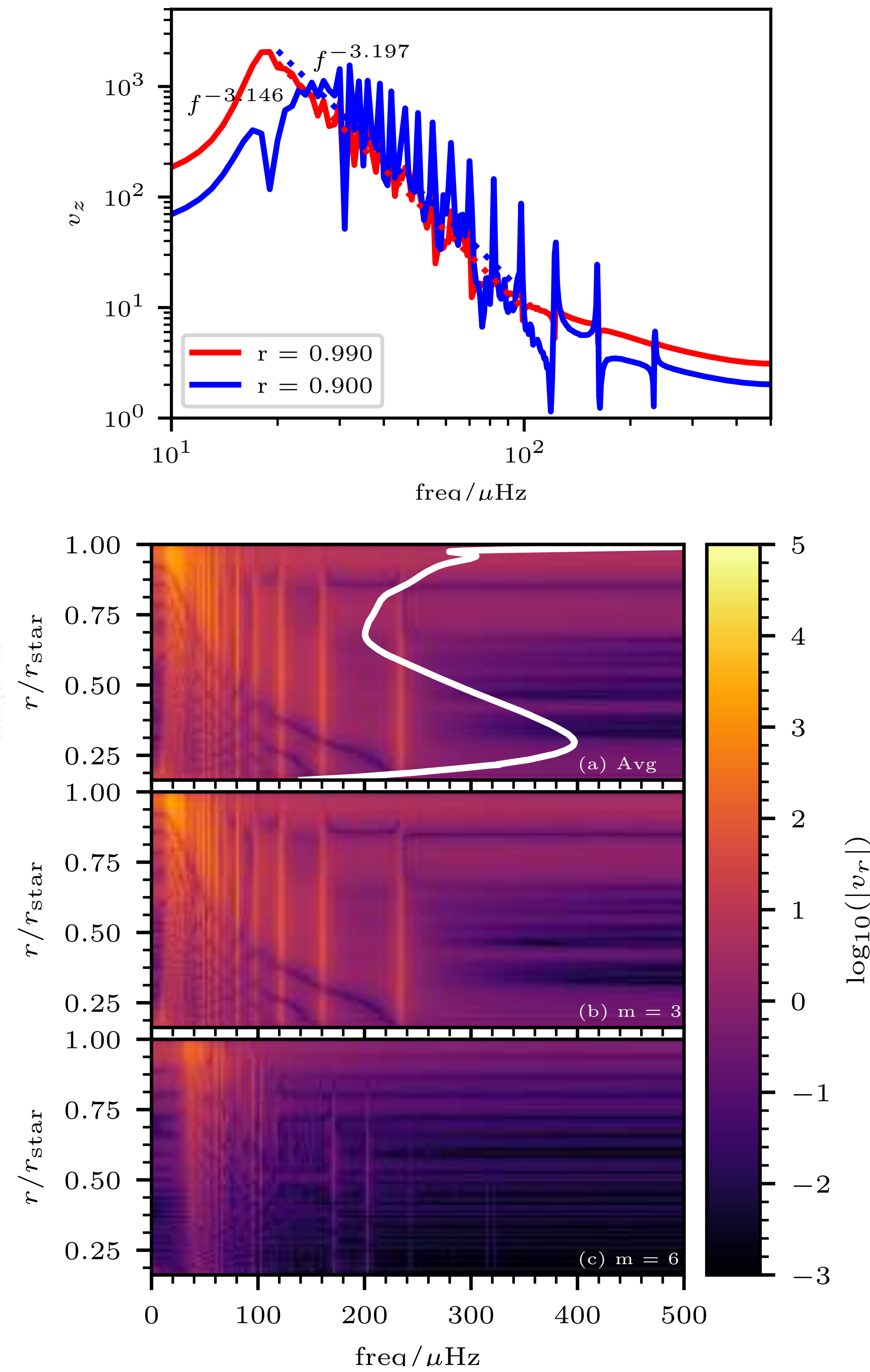
- Solve full set of nonlinear hydrodynamic equations in the anelastic approximation with the stellar thermal diffusivity and Brunt-Vaisala frequency (viscosity still too high)

- Artificially force waves at the bottom boundary to simulate stellar convection (see next lecture for self-consistent calculations)

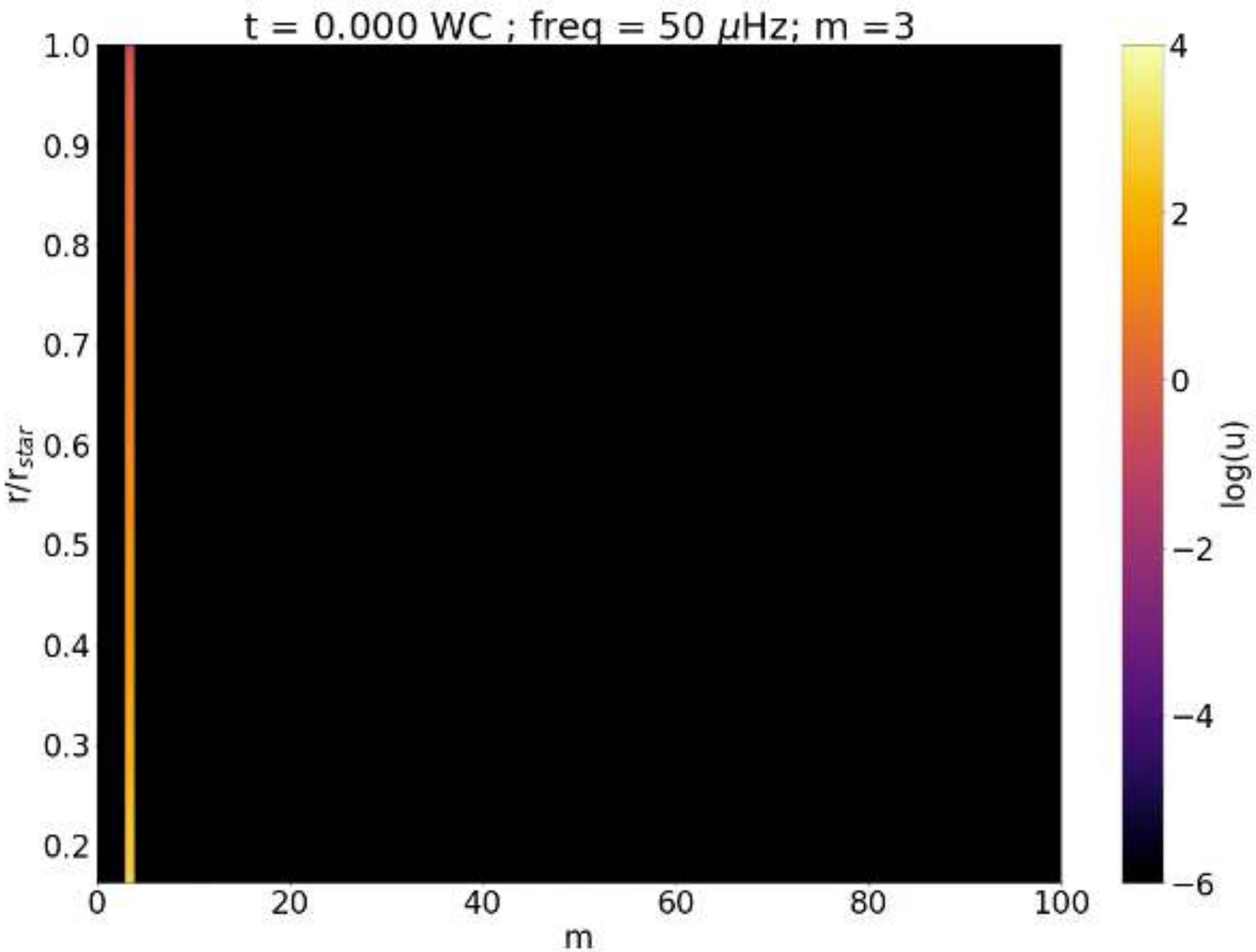


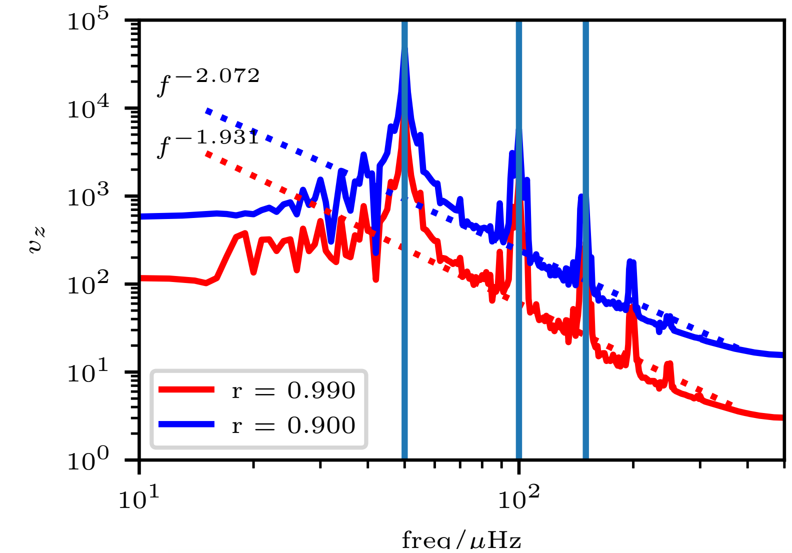
Forcing a single wave in a massive star

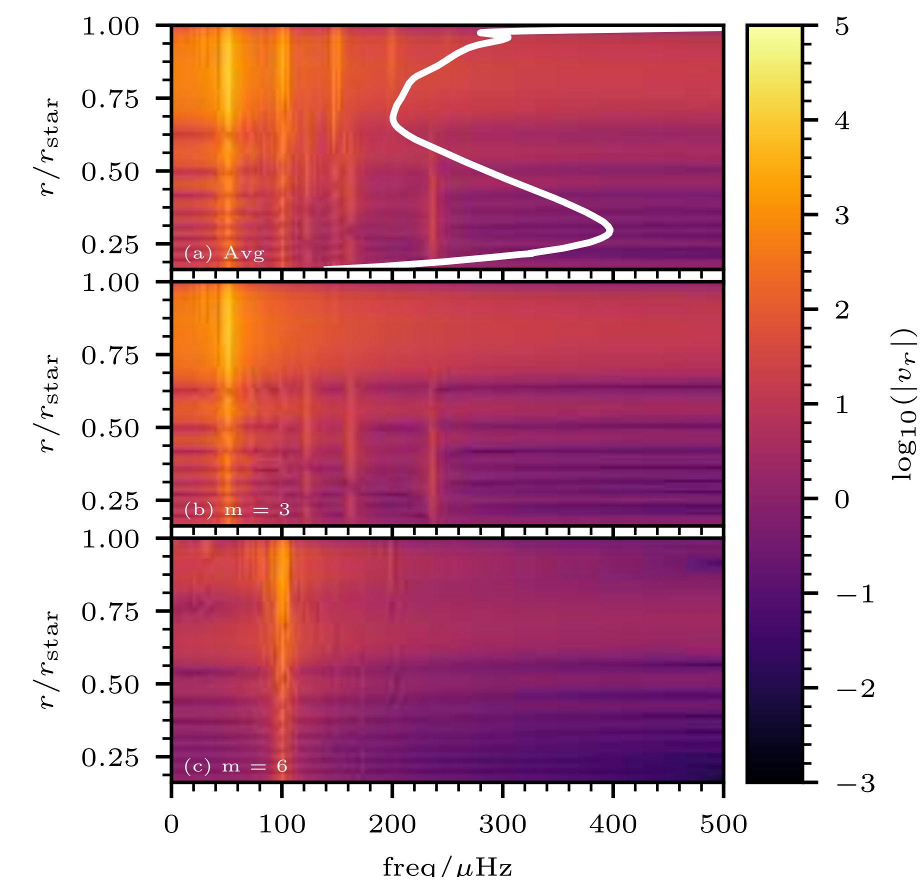




Forcing a single wave in a massive star







Forcing a spectrum of waves in a massive star