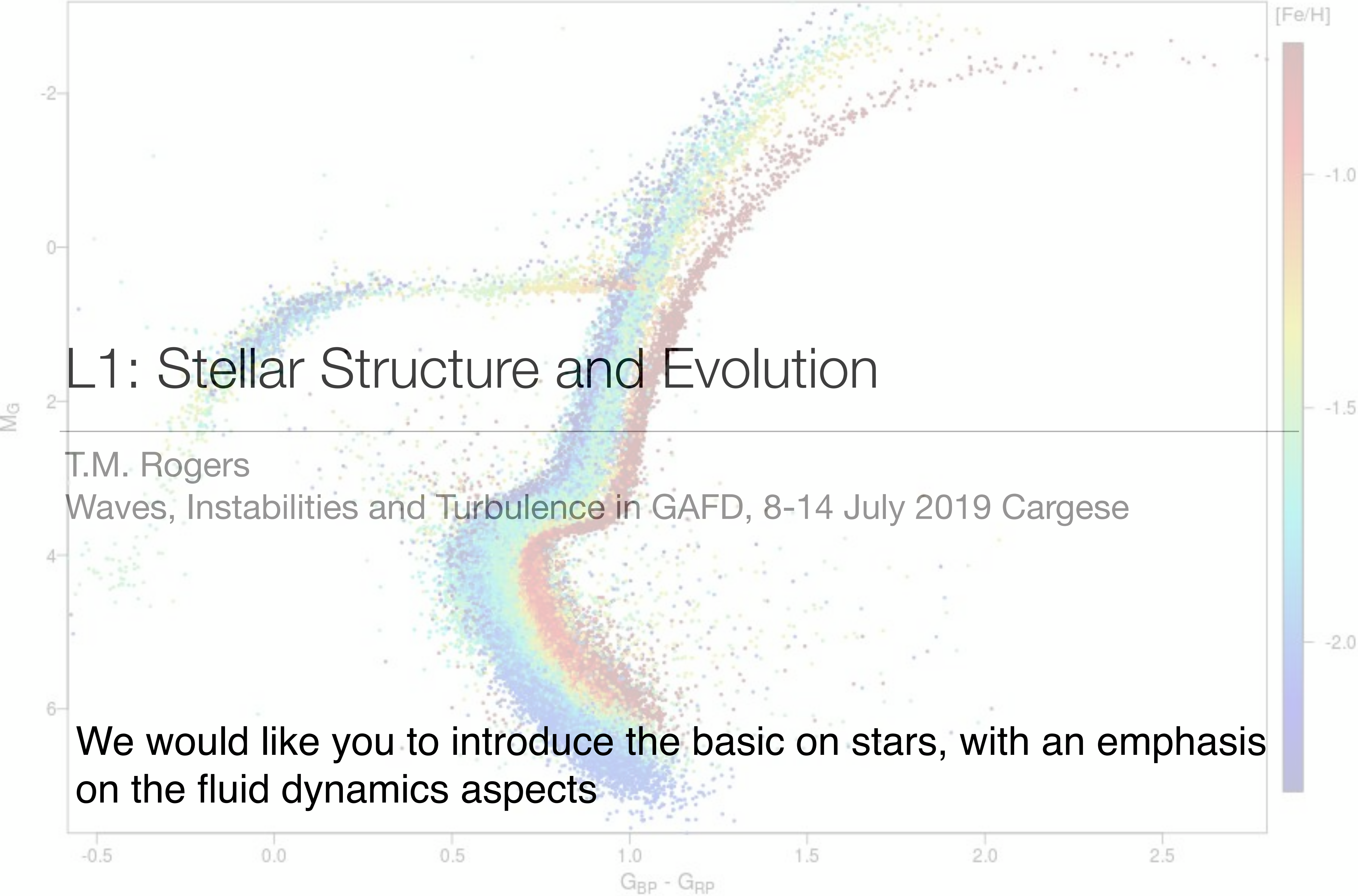
Compressible Convection Conference 2019

- Theme: Compressible waves, instabilities and turbulence in stellar and planetary interiors and atmospheres
- Sept 2-6 2019 in Newcastle, UK
- Registration deadline extended until 2 August
- https://conferences.ncl.ac.uk/ccc2019/



L1: Stellar Structure and Evolution

T.M. Rogers
Waves, Instabilities and Turbulence in GAFD, 8-14 July 2019 Cargese

We would like you to introduce the basic on stars, with an emphasis on the fluid dynamics aspects

Outline:

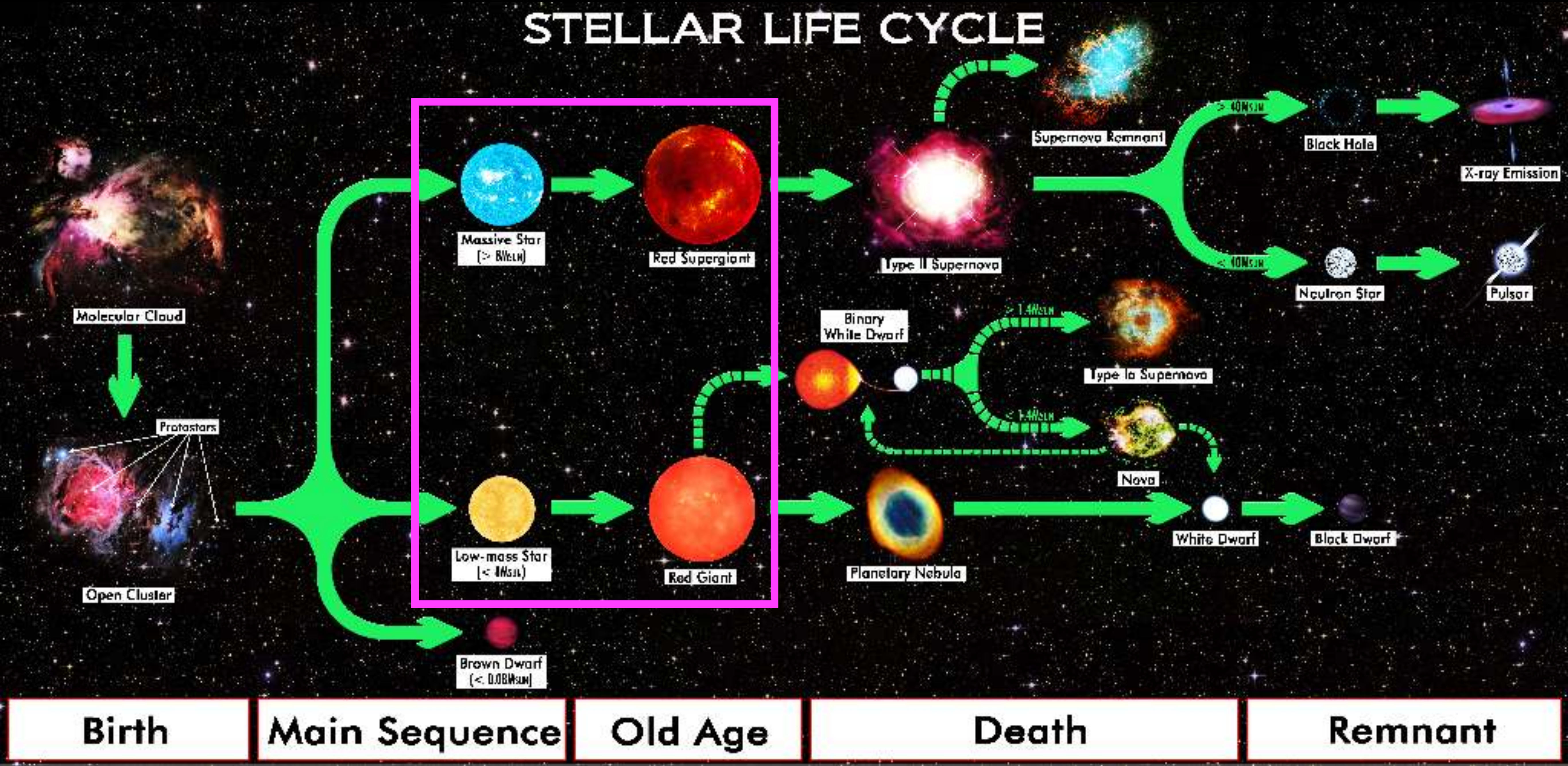
L1: Stellar Structure and Evolution (Theory)
L2: Observations of Stars (aka: how we can constrain the theory)
L3: Basics of Angular Momentum transport by IGW in stellar interiors
L4: Simulations of AM xport and mixing by IGW in massive stars

Cant really do experiments but can look at many stars at
different ages, masses, composition, rotation

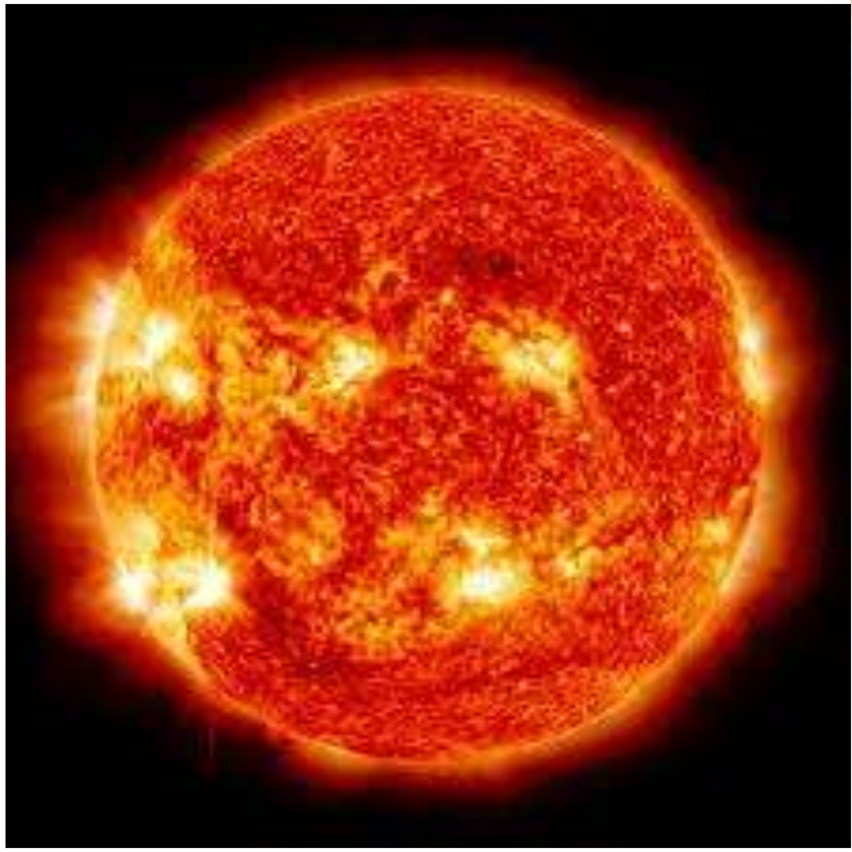
To a high degree of accuracy stars are spheres
Since we cant spatially resolve them (generally) we can treat
them as 1D, generally without rotation

The equations of stellar structure and evolution are
that of mass, momentum and energy conservation along with
an equation of state and composition evolution

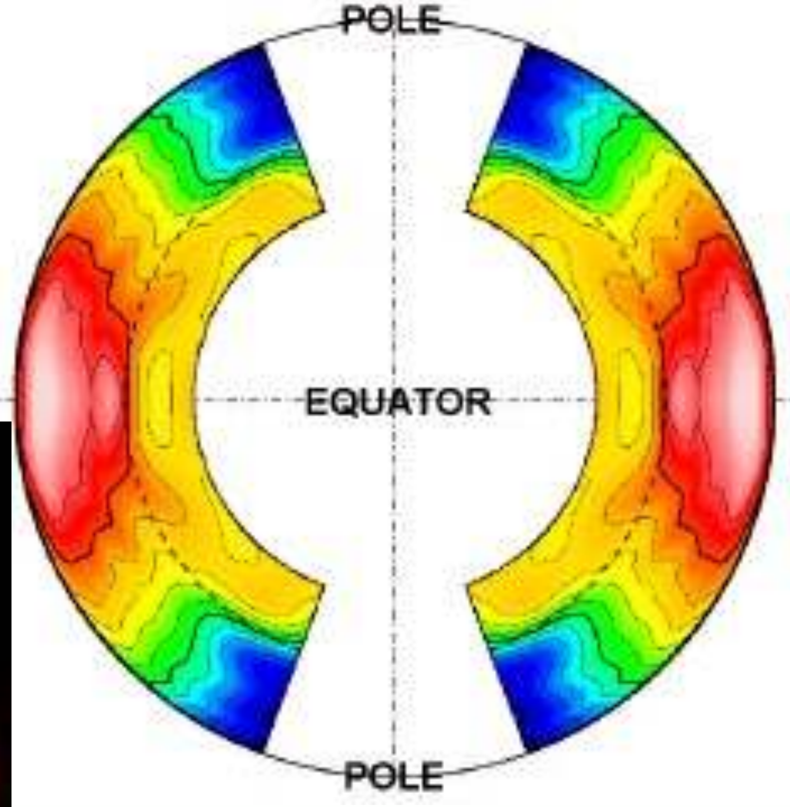
STELLAR LIFE CYCLE



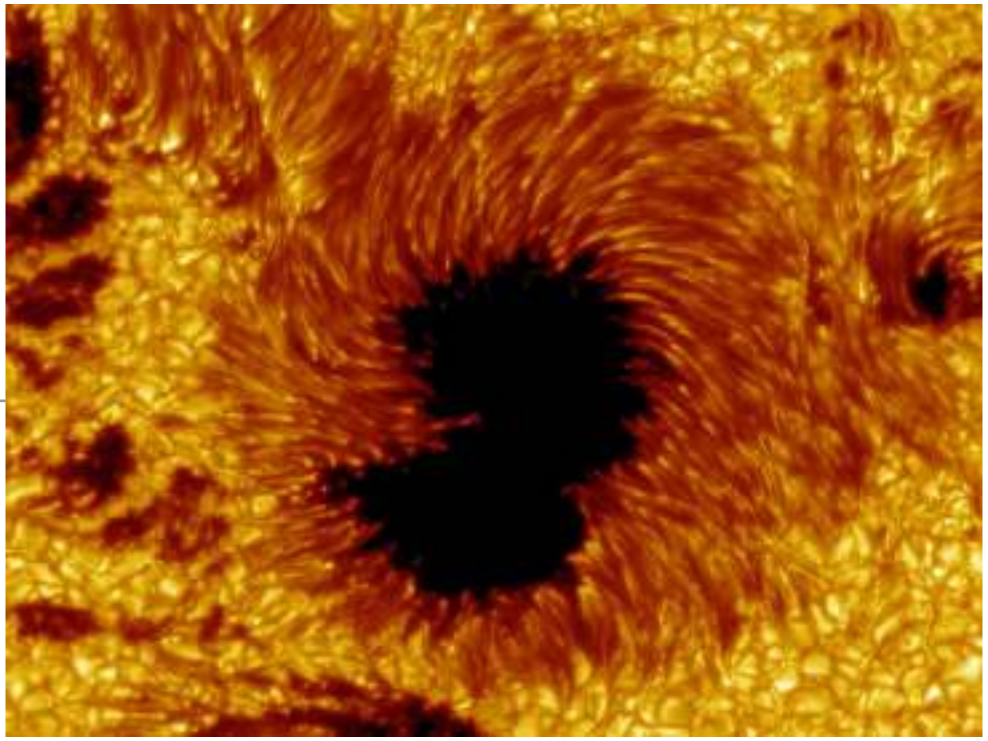
Timescales



Age~5x10^{9} yr
will “live” 10^{10} yr

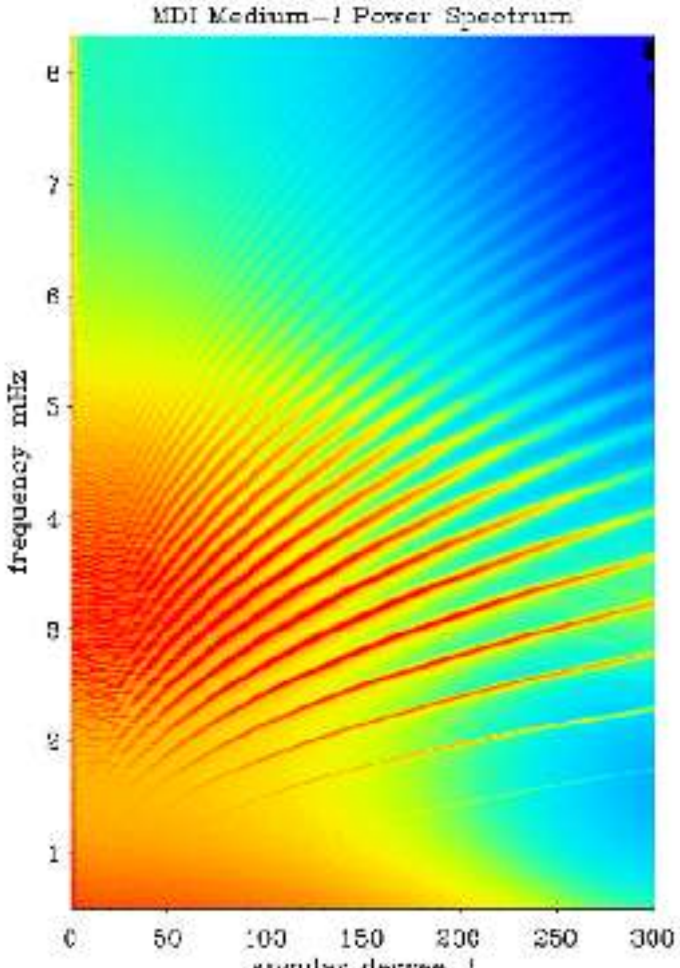


Rotation ~
1/month



Magnetic cycle ~11 yr
Sunspot life ~days-weeks

p-modes
~5 min
g-modes
~10 days



Can’t simulate all these effects at once

Non-dimensional numbers

$$Re = \frac{UL}{\nu} = \frac{(10^2)(10^8)}{10^{-4}} \sim 10^{14}$$

Even when we simulate for short periods of time we can't use correct parameters

$$Ro = \frac{U}{2\Omega L} = \frac{10^2}{(10^{-6})(10^8)} \sim 1$$

$$Ek = \frac{\nu}{2\Omega L^2} = \frac{10^{-4}}{(10^{-6})(10^{16})} \sim 10^{-14}$$

So cant simulate long enough or turbulent enough

$$Pr = \frac{\nu}{\kappa} = \frac{10^{-4}}{10^2} \sim 10^{-6}$$

I got 99 problems and…

Equations of Stellar Structure and Evolution

Kippenhahn & Weigert

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} \ , \tag{10.1}$$

Mass Conservation

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \ , \tag{10.2}$$

Momentum Conservation
(Hydrostatic Equilibrium)

$$\frac{\partial l}{\partial m} = \varepsilon_{\mathrm{n}} - \varepsilon_{\nu} \tag{10.3}$$

Energy Conservation

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla \ , \qquad \nabla = \nabla_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4} \ . \tag{10.4}$$

Energy Transport

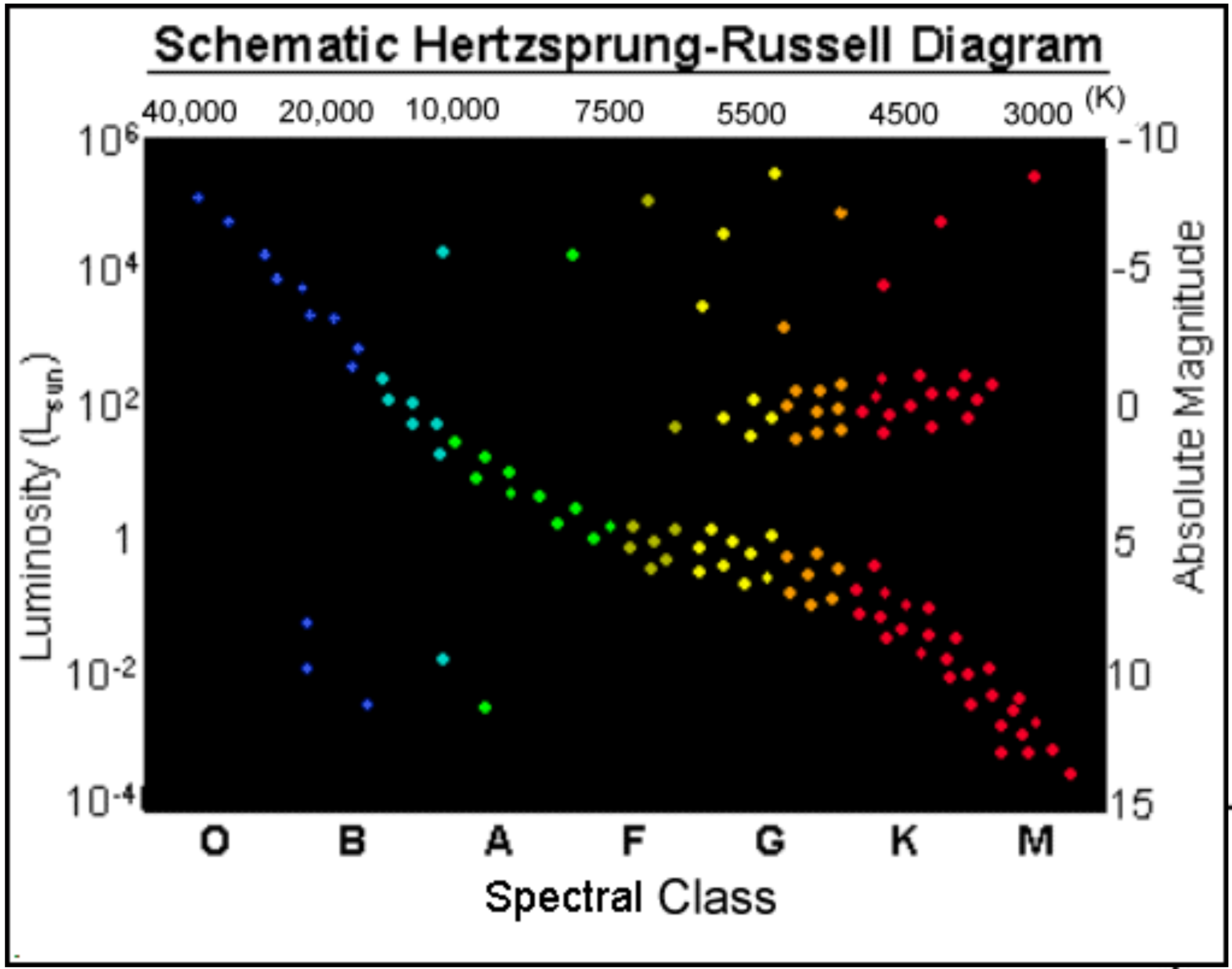
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) \ , \quad i = 1, \ldots, I \ . \tag{10.5}$$

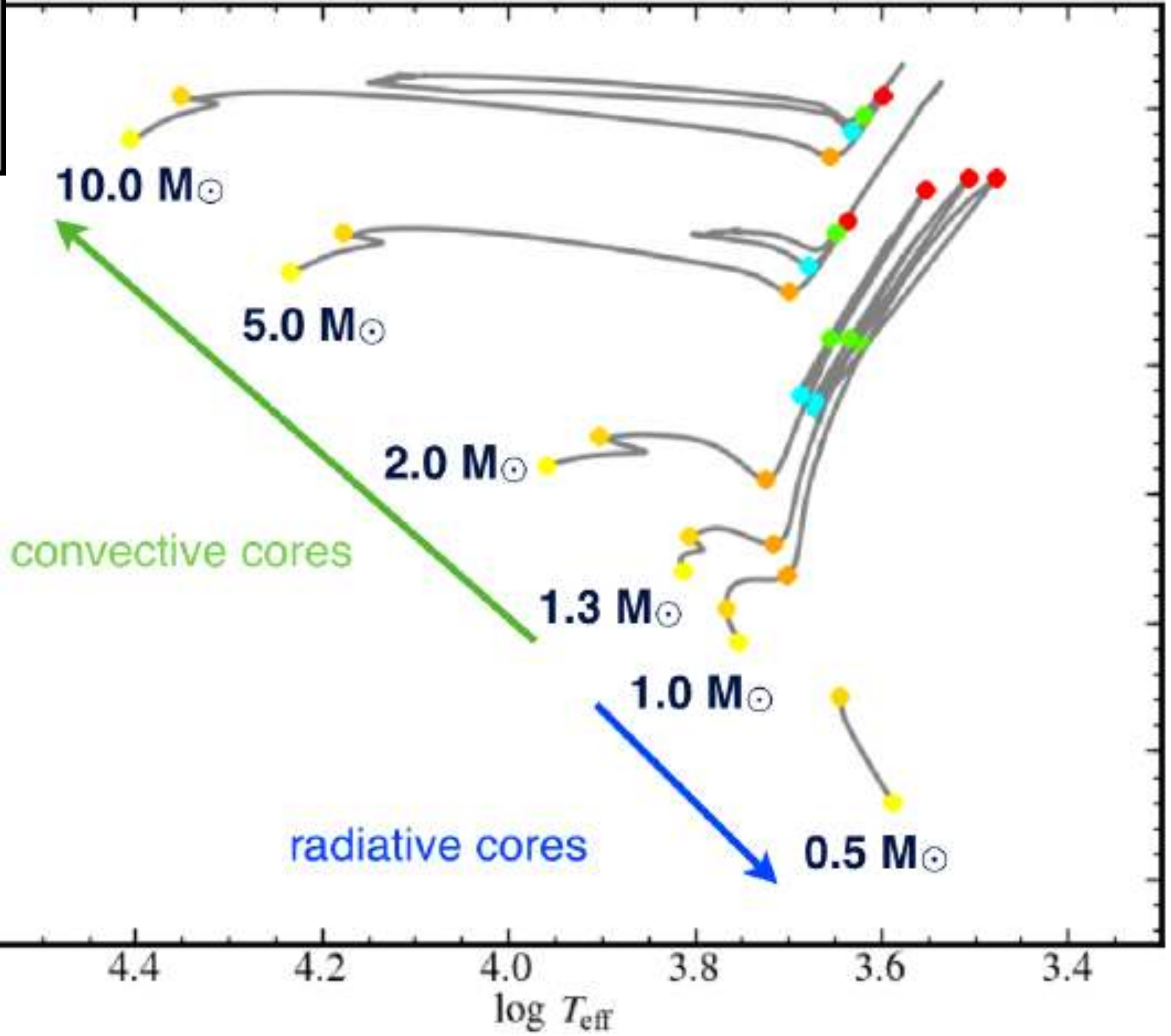
Chemical Evolution

+ Equation of State
(simplest version IGL) **+** Opacity
(simplest version Kramer's) **+** Energy Generation

$$\epsilon_{pp} \simeq A\rho X^2 T_6^4 \qquad \epsilon_{CNO} \simeq B\rho X X_{CNO} T_6^{20} \qquad \epsilon_{3\alpha} \simeq C\rho^2 Y^3 T_8^{41}$$

HR Diagram


Schematic Hertzsprung-Russell Diagram
40,000
20,000
10,000
7500
5500
4500
3000
(K)
10^6
10^4
10^2
1
10^{-2}
10^{-4}
Luminosity (L_{sun})
-10
-5
0
5
10
15
Absolute Magnitude
O
B
A
F
G
K
M
Spectral Class


10.0 M⊙
5.0 M⊙
2.0 M⊙
1.3 M⊙
1.0 M⊙
0.5 M⊙
convective cores
radiative cores
3
2
1
0
−1
−2
log $L/L_\odot$
4.4
4.2
4.0
3.8
3.6
3.4
log $T_{\rm eff}$

Momentum Equation/Hydrostatic Equilibrium

$$\cancel{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}} = \mathbf{g} - \cancel{\mathbf{2}\,(\boldsymbol{\Omega}\times\mathbf{v})} - \frac{\nabla \mathbf{P}}{\bar{\rho}} + \cancel{\frac{\mathbf{1}}{\mu_{\mathbf{o}}\bar{\rho}}(\nabla\times\mathbf{B})\times\mathbf{B}} + \cancel{\bar{\nu}\nabla^{2}\mathbf{v}}$$

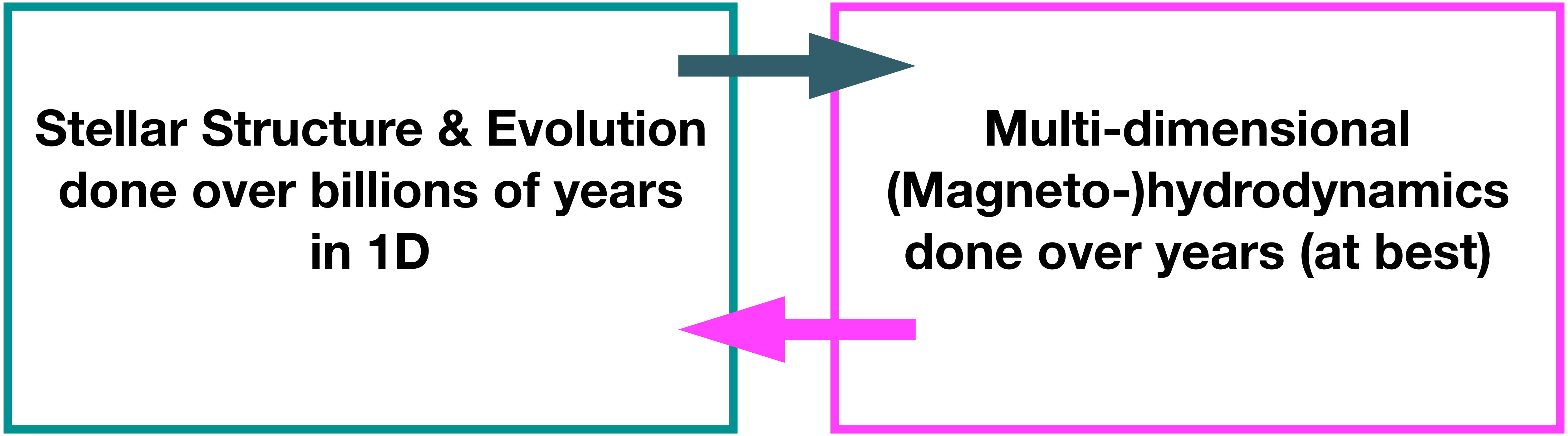
Steady State

No rotation

No Magnetism

Inviscid

$$\nabla P = -\bar{\rho}\mathbf{g}$$ Hydrostatic Equilibrium



Stellar Structure & Evolution done over billions of years in 1D

Multi-dimensional (Magneto-)hydrodynamics done over years (at best)

Energy Transport

- Region is deemed to be convective/ radiative by the Schwarzschild criteria (no chemical gradients) or the Ledoux criteria (including chemical gradients)

- If convective - use Mixing Length Theory to calculate a diffusion coefficient. This is a very high number that ensures complete mixing of species and angular momentum within convection zones

- If radiative - employ theoretical predictions for mixing at convective-radiative interfaces (overshoot), for mixing by various physical instabilities and circulation (which physical instabilities if no rotation?)

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla$$

If sub-adiabatic (radiative):

$$\nabla = \nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4} \ .$$

If super-adiabatic (convective):

$$\nabla_{\rm ad} \equiv \left(\frac{P}{T}\frac{dT}{dP}\right)_s$$

Convective Instability & Internal Gravity Waves

$\rho_a(z_0+h)$

$\rho_p(z_0+h) \neq \rho_a(z_0+h)$

if $\rho_p(z_0+h) < \rho_a(z_0+h)$ parcel continues to rise $\Rightarrow$ <u>convection</u>

displace parcel upward <u>adiabatically</u>
(loses no heat, so $P_p = P_{atm}$)
parcel moves adiabatically

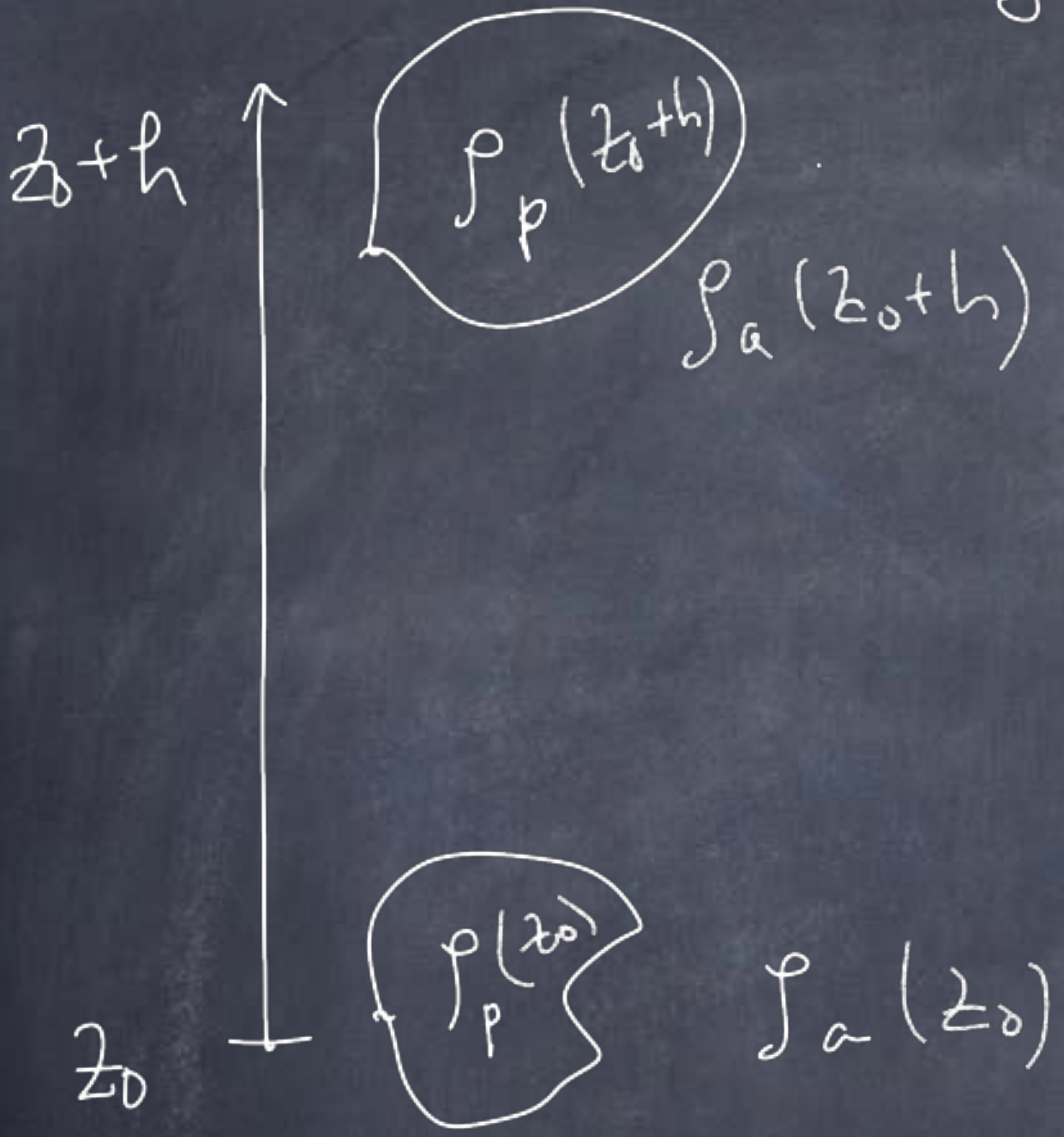
parcel of air w/ density
ρ_p, background / atmosphere density
$= \rho_a$

$\rho_a(z_0)$

$\rho_p(z_0) = \rho_a(z_0)$ $P_p(z_0) = P_a(z_0)$

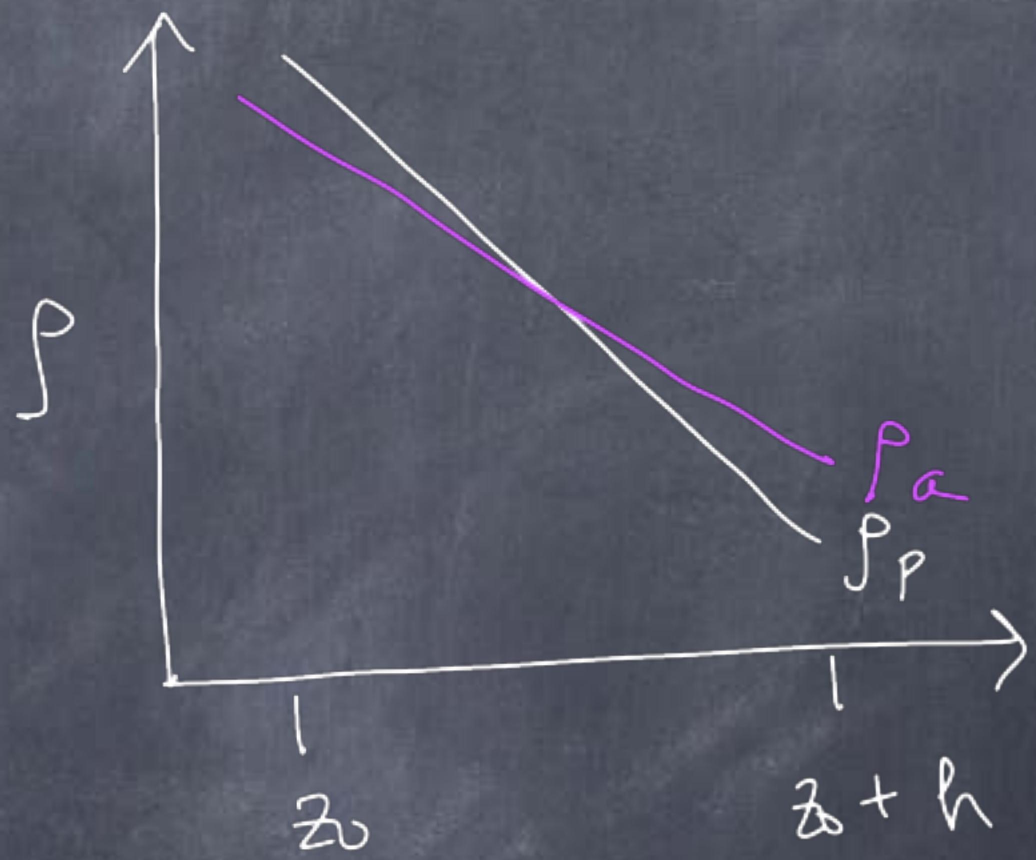
$T_p(z_0) = T_a(z_0)$

When convectively unstable (super-adiabatic)



$\rho_p < \rho_a$ ⇙ IGL $P \sim R\rho T$

$T_p > T_a$



We call this super-adiabatic because the temp gradient of the atmosphere is steeper than the adiabatic (parcel) gradient

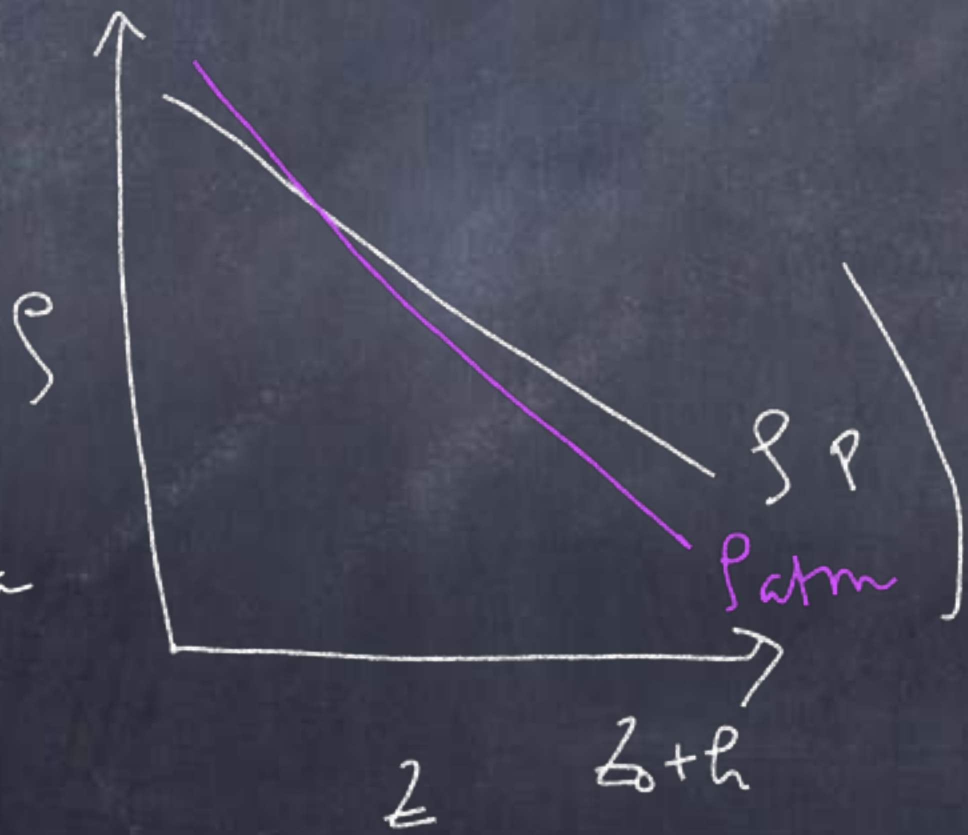
when convectively stable (sub-adiabatic)

$\rho_p > \rho_a$

$T_p < T_a$

parcel sinks back down, but of course, doesnt stop @ z_0 but oscillates about z_0 like a pendulum

= <u>sub</u> adiabatic



$$\rho_p(z_0+h) \neq \rho_a(z_0+h) \quad \& \quad T_p(z_0+h) \neq T_a(z_0+h)$$

if h is small do Taylor expansion

$$\rho_p(z_0+h) = \rho_p(z_0) + \frac{d\rho_p}{dz} h$$

(Not mathematically rigorous)

F = ma

$$-g\left(\rho_p(z_0+h) - \rho_a(z_0+h)\right) \simeq \rho_a(z_0)\frac{d^2h}{dt^2}$$

$$= -g\left(\cancel{\rho_p(z_0)} + \frac{d\rho_p}{dz}h - \cancel{\rho_a(z_0)} - \frac{d\rho_a}{dz}h\right) \simeq \rho_a(z_0)\frac{d^2h}{dt^2}$$

@ z_0 $\rho_p = \rho_a$

$$\Rightarrow -g\left(\frac{d\rho_p}{dz} - \frac{d\rho_a}{dz}\right)h \simeq \rho_p(z_0)\frac{d^2h}{dt^2}$$

$$\Rightarrow \frac{d^2h}{dt^2} + \frac{g}{\rho_p(z_0)}\left(\frac{d\rho_p}{dz} - \frac{d\rho_a}{dz}\right)h = 0.$$

$$\frac{d^2h}{dt^2} + N^2 h = 0.$$

$$N^2 = \frac{g}{\rho}\left(\frac{d\rho_p}{dz} - \frac{d\rho_a}{dz}\right)h$$

= Brunt-Vaisala Frequency

For Ideal Gas in H = m

$$T\left(\frac{d\rho_p}{dz} - \frac{d\rho_a}{dz}\right) = \rho\left(\frac{dT_a}{dz} - \frac{dT_p}{dz}\right)$$

$$N^2 = \frac{g}{\cancel{\rho}}\frac{\cancel{\rho}}{T}\left(\frac{dT_a}{dz} - \frac{dT_p}{dz}\right)$$

$N^2 > 0 \Rightarrow$

$$\frac{g}{T}\left(\frac{dT_a}{dz} - \frac{dT_p}{dz}\right) > 0$$ which means

$$\left|\frac{dT_p}{dz}\right| > \left|\frac{dT_a}{dz}\right|$$

atmosphere is <u>sub</u> adiabatic

i.e. atmospheric gradient is less steep than adiabatic gradient

$N^2 > 0 \Rightarrow$ Internal Gravity Waves

$N^2 < 0 \Rightarrow$ Convection

Better Way

Start w/ Boussinesq Equations:

$$\nabla \cdot \vec{V} = 0$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\nabla P - \frac{\rho'}{\rho_0} g \hat{z} + \nu \nabla^2 \vec{V}$$

$$\frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T = \gamma K \nabla^2 T$$

For a liquid (Bouss)

$$\rho' = -\rho_0 \alpha T'$$

$$\Rightarrow \frac{d\vec{V}}{dt} = -\nabla P + \alpha T' g \hat{z} + \nu \nabla^2 \vec{V}$$

$$\frac{dT'}{dt} = \gamma K \nabla^2 T'$$

Assume 2D Cartesian flow in x & z. Gravity in $\hat{z}$ direction.

since $\nabla \cdot v = 0$, can write $\vec{v} = \nabla \times \psi$ then incompressibility ensured exactly

$$v_x = -\frac{\partial \varphi}{\partial z} \qquad v_z = \frac{\partial \varphi}{\partial x}$$

define Vorticity $\vec{\omega} = \nabla \times \vec{v}$

$$= \nabla \times (\nabla \times \psi)$$

$$= -\nabla^2 \psi$$

$$= -\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}\right)$$

Now take $\nabla \times$ of M.E.

$$\nabla \times \left\{ \frac{d\vec{v}}{dt} = -\nabla P + \alpha T' g \hat{z} + \nu \nabla^2 v \right\}$$

$$\Rightarrow \quad \frac{d\omega}{dt} = \underbrace{\nabla \times \left(\alpha T' g \hat{z} \right)} + \cancel{\nu \nabla^2 \omega}$$

neglect
viscosity

$$\frac{d\omega}{dt} = -\alpha g \frac{\partial T}{\partial x}$$

Now linearize

$$+ \quad \frac{dT}{dt} = 0. \quad \left(\begin{array}{c} \text{neglect} \\ \text{diffusion} \end{array} \right)$$

$$\omega = \omega'$$

$$T = \bar{T}(z) + T'$$

$$\frac{\partial \omega'}{\partial t} + \cancel{v \cdot \nabla \omega'} = -\alpha g \left(\cancel{\frac{\partial \bar{T}}{\partial x}} + \frac{\partial T'}{\partial x} \right)$$

$$\bar{T} = \bar{T}(z)$$

$$\Rightarrow \quad \frac{\partial \omega'}{\partial t} = -\alpha g \frac{\partial T'}{\partial x}$$

$$\frac{\partial T}{\partial t} + (v\cdot\nabla)T = 0$$

$$\cancel{\frac{\partial \bar{T}}{\partial t}} + \frac{\partial T'}{\partial t} + (v\cdot\nabla)(\bar{T}+\cancel{T'}) = 0.$$

NL

$\bar{T} \to$ fn. of z only

$\bar{T}$ fn of z only

so

$$\Rightarrow \frac{\partial T'}{\partial t} + (v\cdot\nabla)\bar{T} = 0$$

$$\frac{\partial T'}{\partial t} + v_z \frac{\partial \bar{T}}{\partial z} = 0.$$

$$+ \frac{\partial w'}{\partial t} + \alpha g \frac{\partial T'}{\partial x} = 0.$$

Take time derivative of vorticity eqn

$$\frac{\partial}{\partial t}\left\{\frac{\partial \omega'}{\partial t} = -\alpha g \frac{\partial T}{\partial x}\right\} \Rightarrow \frac{\partial^2 \omega'}{\partial t^2} = -\alpha g \frac{\partial}{\partial x}\left(\frac{\partial T'}{\partial t}\right)$$

substitute in $\frac{\partial T'}{\partial t} = -v_z \frac{\partial \bar{T}}{\partial z}$

$$\Rightarrow \frac{\partial^2 \omega'}{\partial t^2} = -\alpha g \frac{\partial}{\partial x}\left\{-v_z \frac{\partial \bar{T}}{\partial z}\right\} \qquad v_z = \frac{\partial \varphi}{\partial x}$$

$$= -\alpha g \frac{\partial}{\partial x}\left\{-\frac{\partial \varphi}{\partial x}\right\}\frac{\partial \bar{T}}{\partial z}$$

$$= -\alpha g \frac{\partial^2 \varphi}{\partial x^2} \underbrace{\frac{\partial \bar{T}}{\partial z}}$$

$$\omega = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}\right) = N^2 \frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right)=N^2\frac{\partial^2\psi}{\partial x^2}$$

wave soln: $\psi(x,z,t)=\psi_0\, e^{i(k_x x+k_z z-\omega t)}$

$\uparrow$ frequency

NOT vorticity

$$\Rightarrow (k_x^2+k_z^2)\,\omega^2\,\cancel{\psi_0 e^{i(\;)}} = N^2 k_x^2\,\cancel{\psi_0 e^{i(\;)}}$$

$$\omega^2=\frac{N^2k_x^2}{(k_x^2+k_z^2)}$$

$N^2>0$ waves

$N^2<0$ decaying

generalized to 3D:

$$\omega^2=\frac{N^2k_x^2}{(k_x^2+k_y^2+k_z^2)}$$

This is same N as before:

Back to 2D:

$$\omega^2 = \frac{N^2 k_x^2}{(k_x^2 + k_z^2)} = N^2 \frac{k_x^2}{|\underline{k}|^2} \Rightarrow$$

$$\frac{\omega^2}{N^2} = \frac{k_x^2}{|\underline{k}|^2} \rightarrow$$

between 0 & 1

when = 1 there is no vertical component

so $0 < \omega < N$

for IGW.

phase speed $C_p = (C_{px}, C_{pz}) = \left(\frac{\omega}{k_x}, \frac{\omega}{k_z}\right)$

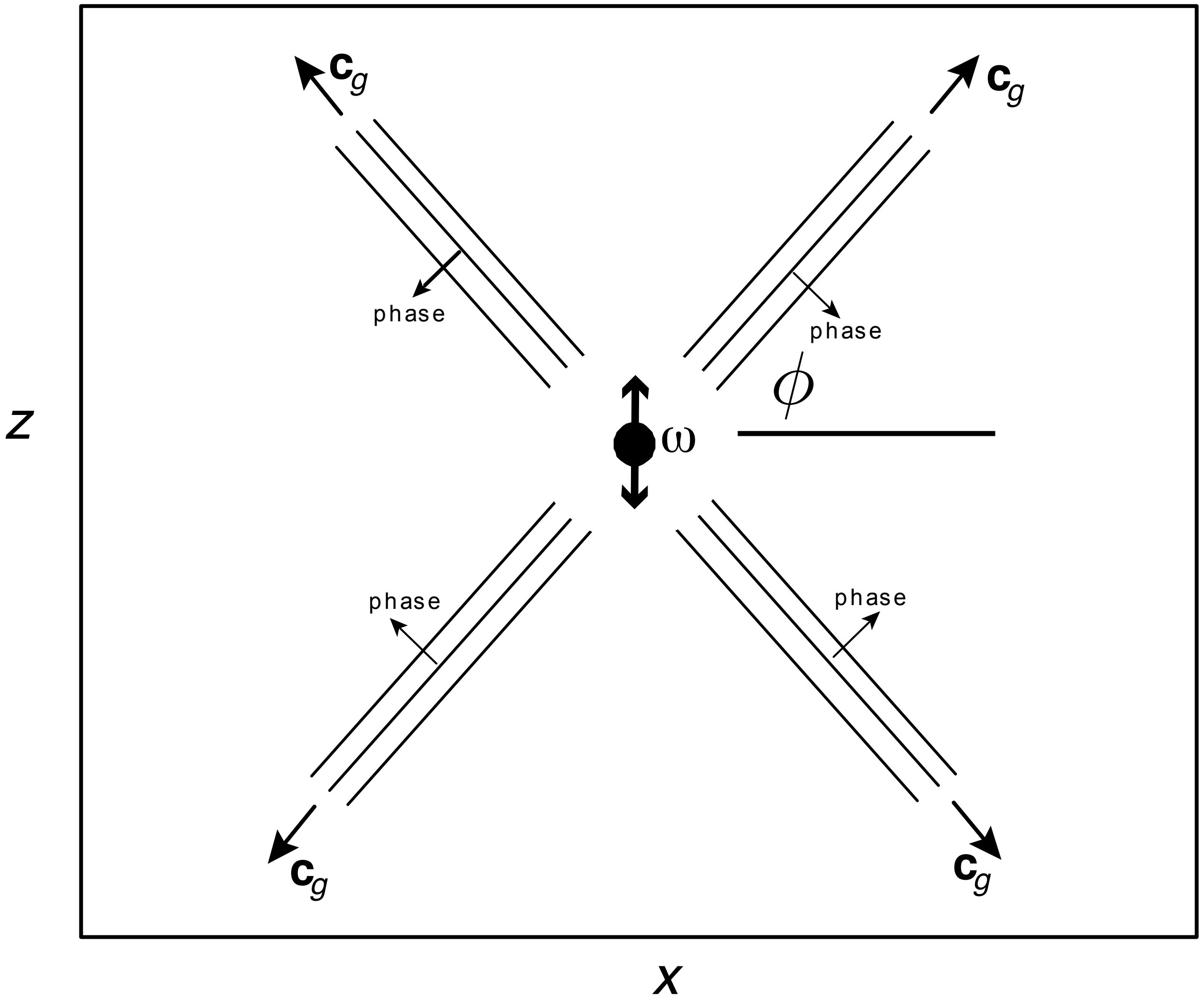
$$= \frac{N k_x}{(k_x^2+k_z^2)^{3/2}} \left(k_x \hat{x} + k_z \hat{z}\right)$$

group velocity $= C_g = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_z}\right)$

$$= \frac{N k_z}{(k_x^2+k_z^2)^{3/2}} \left(k_z \hat{x} - k_x \hat{z}\right)$$

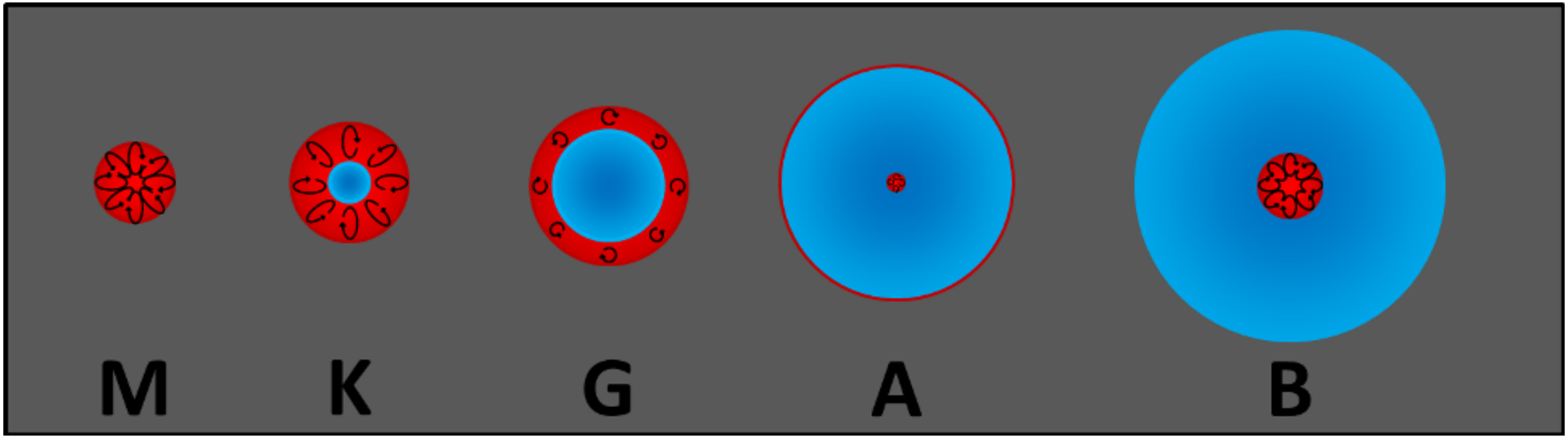
vertical components in opposite directi

Dispersion Relation

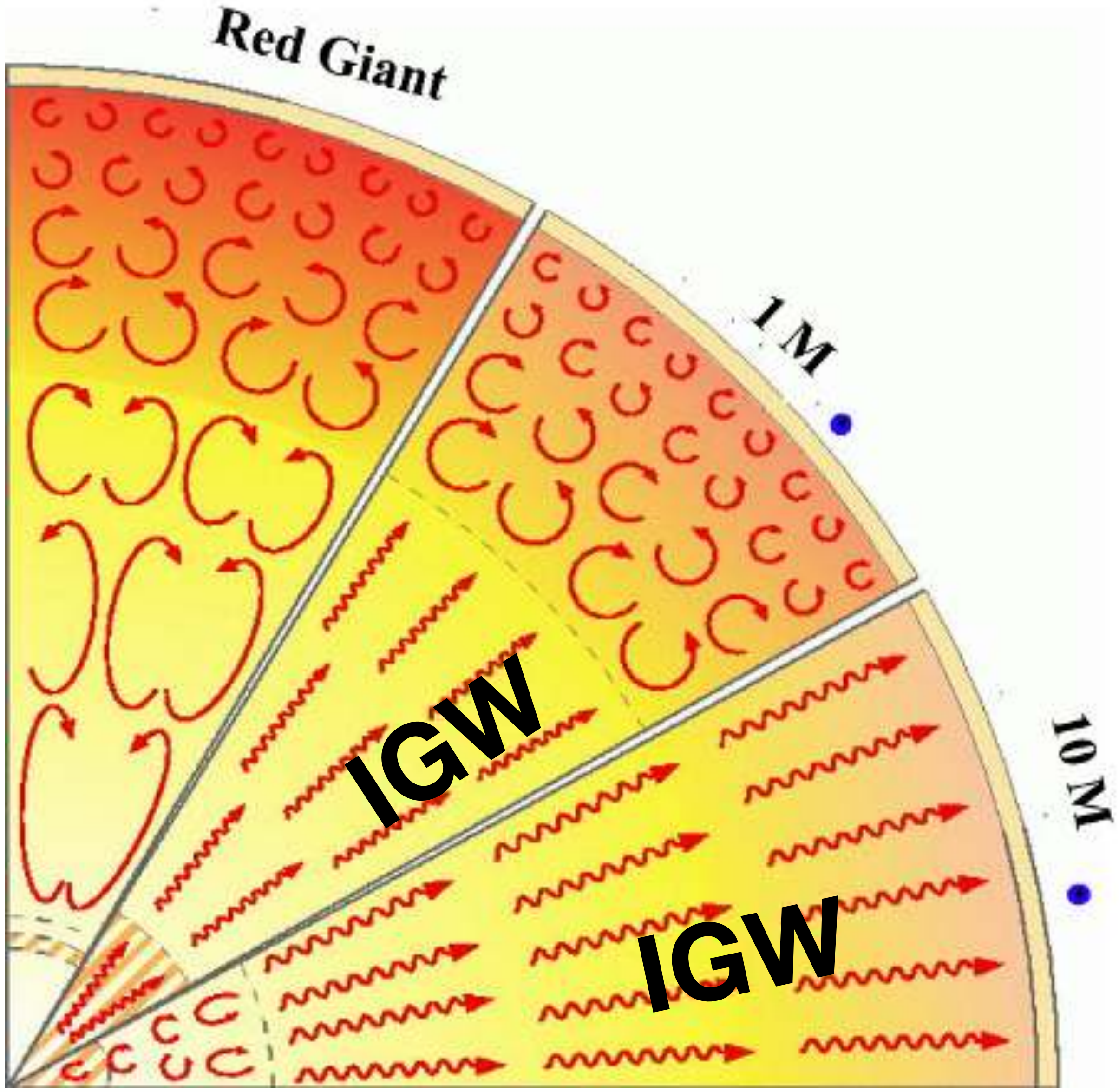


$$\omega^2 = \frac{k_x^2 N^2}{k_x^2 + k_z^2}$$

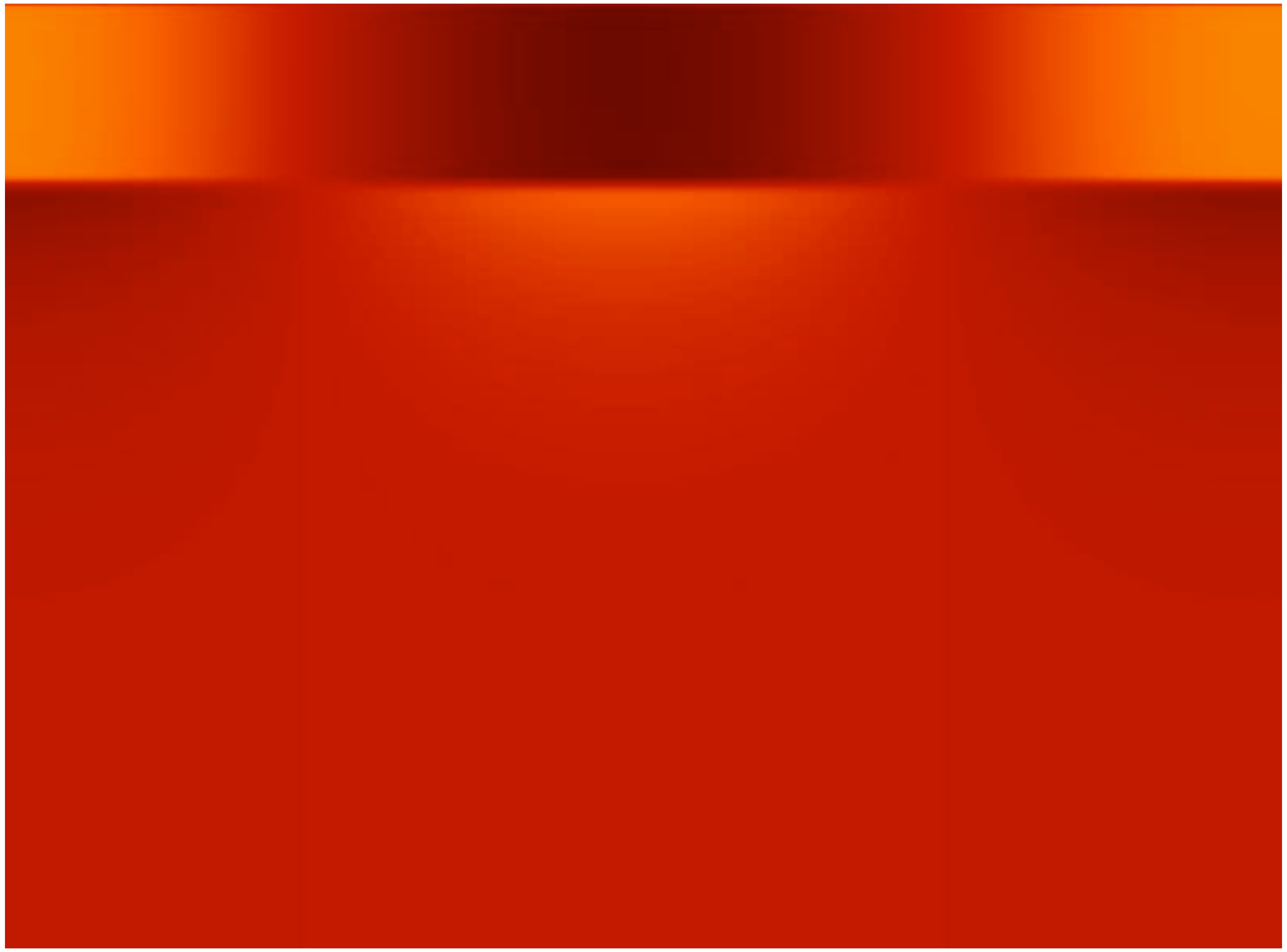
$$\omega = \pm N \sin\phi$$

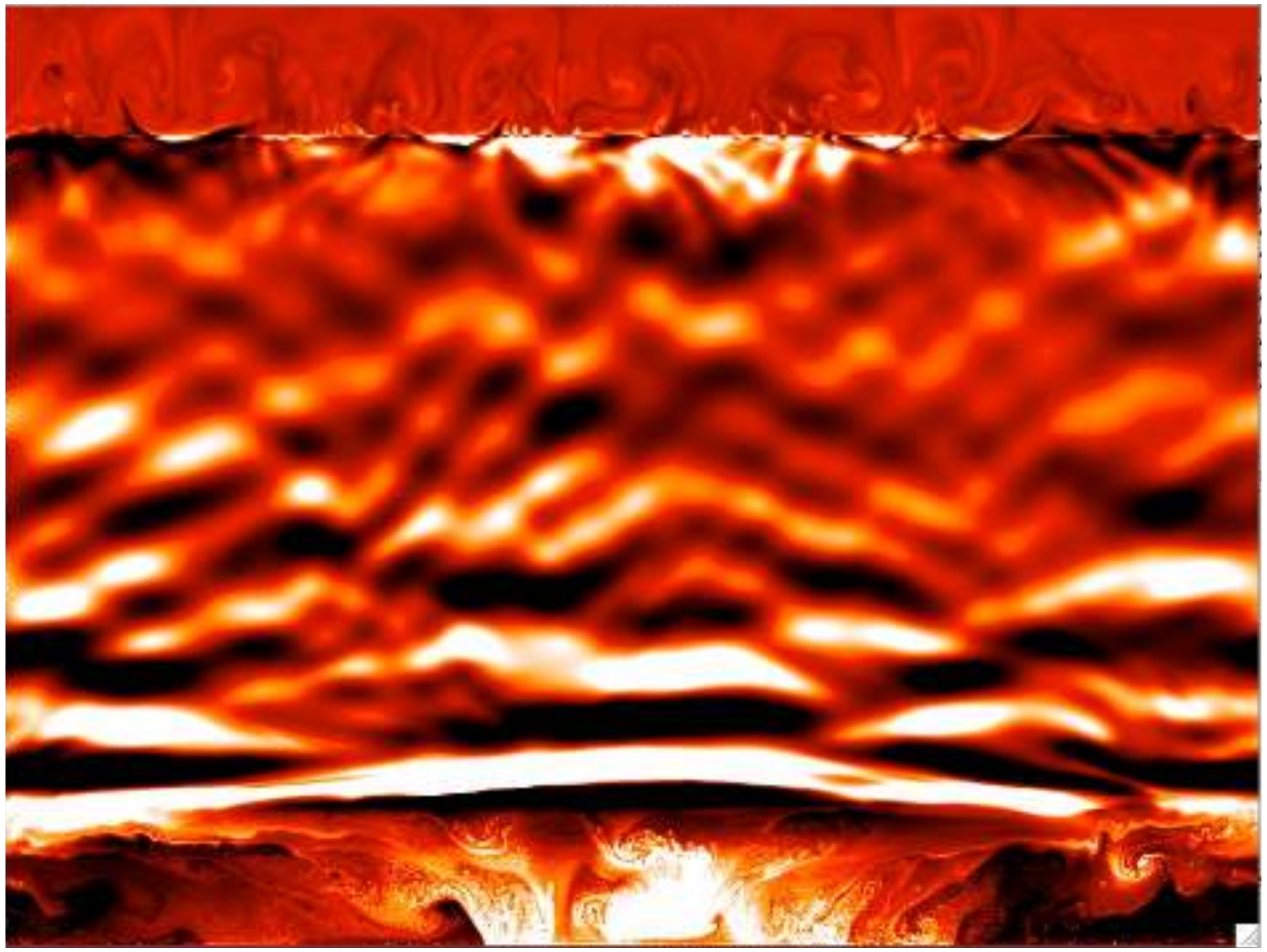


Stars of different masses
(and ages) have fundamentally
different structure
and therefore
different regions where IGW can
propagate and where
instabilities and turbulence occur



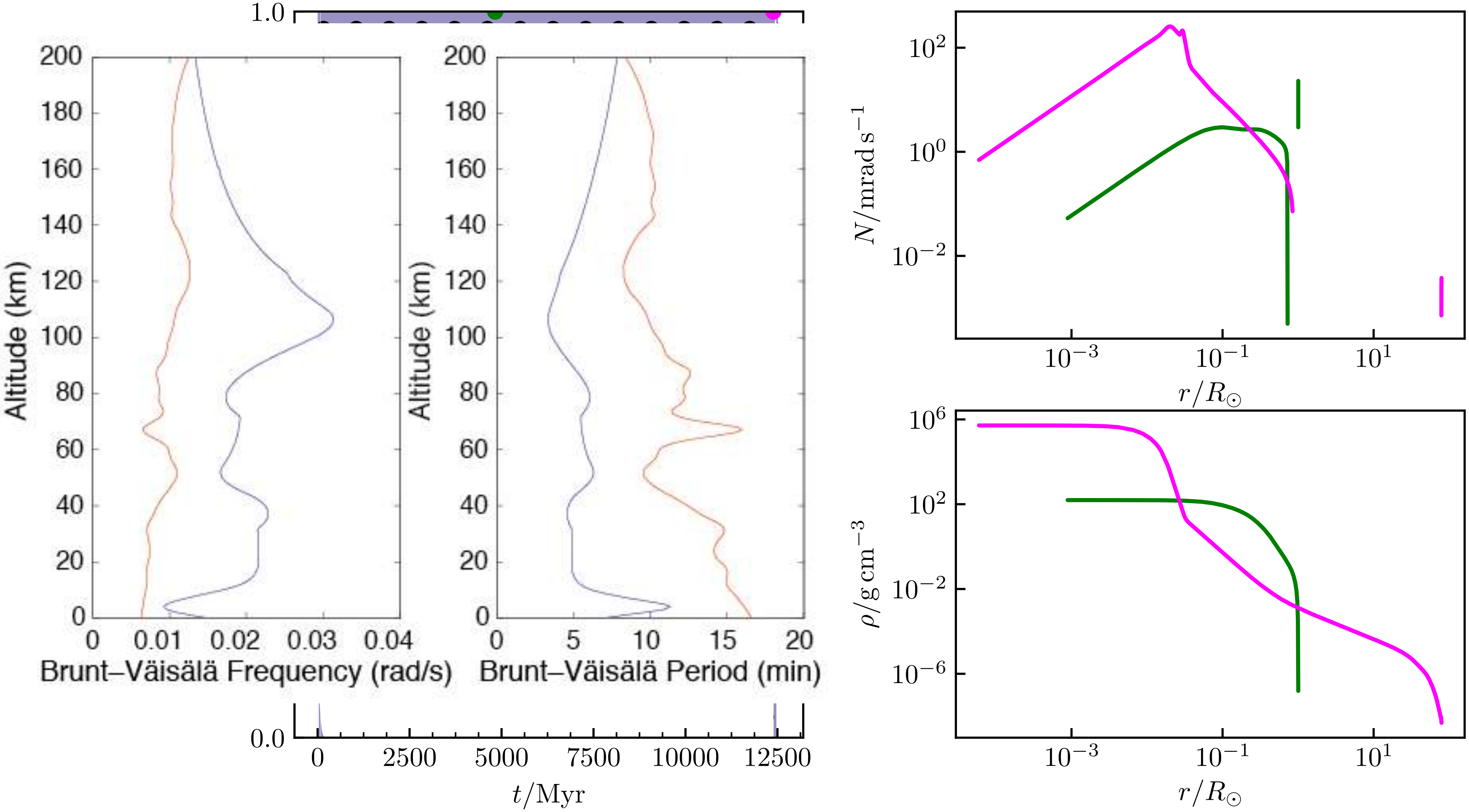
Wave Generation by Convection





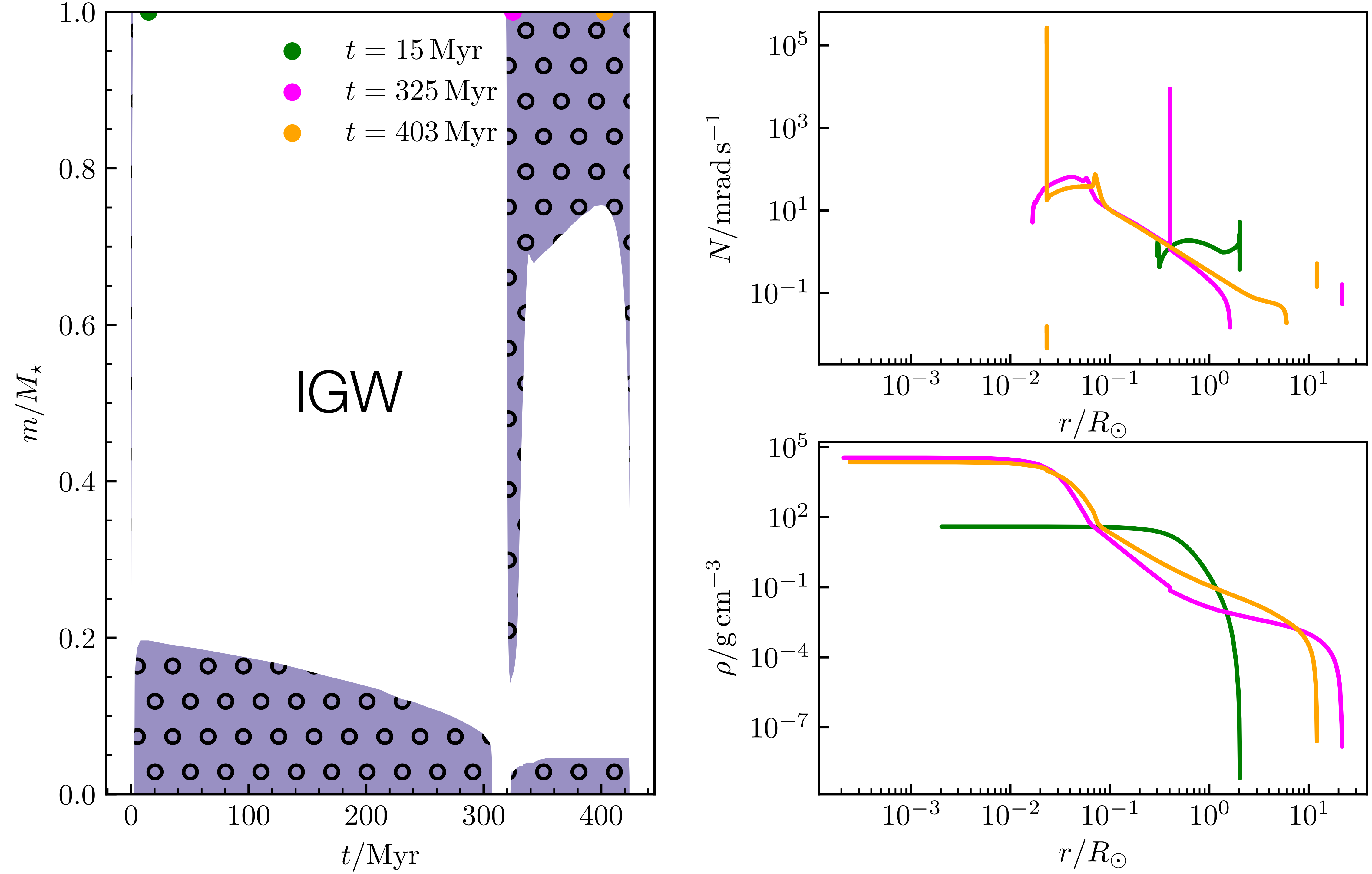
Interior Structure of Stars

$M_{\star,\mathrm{ZAMS}} = 1.0\,\mathrm{M}_\odot$



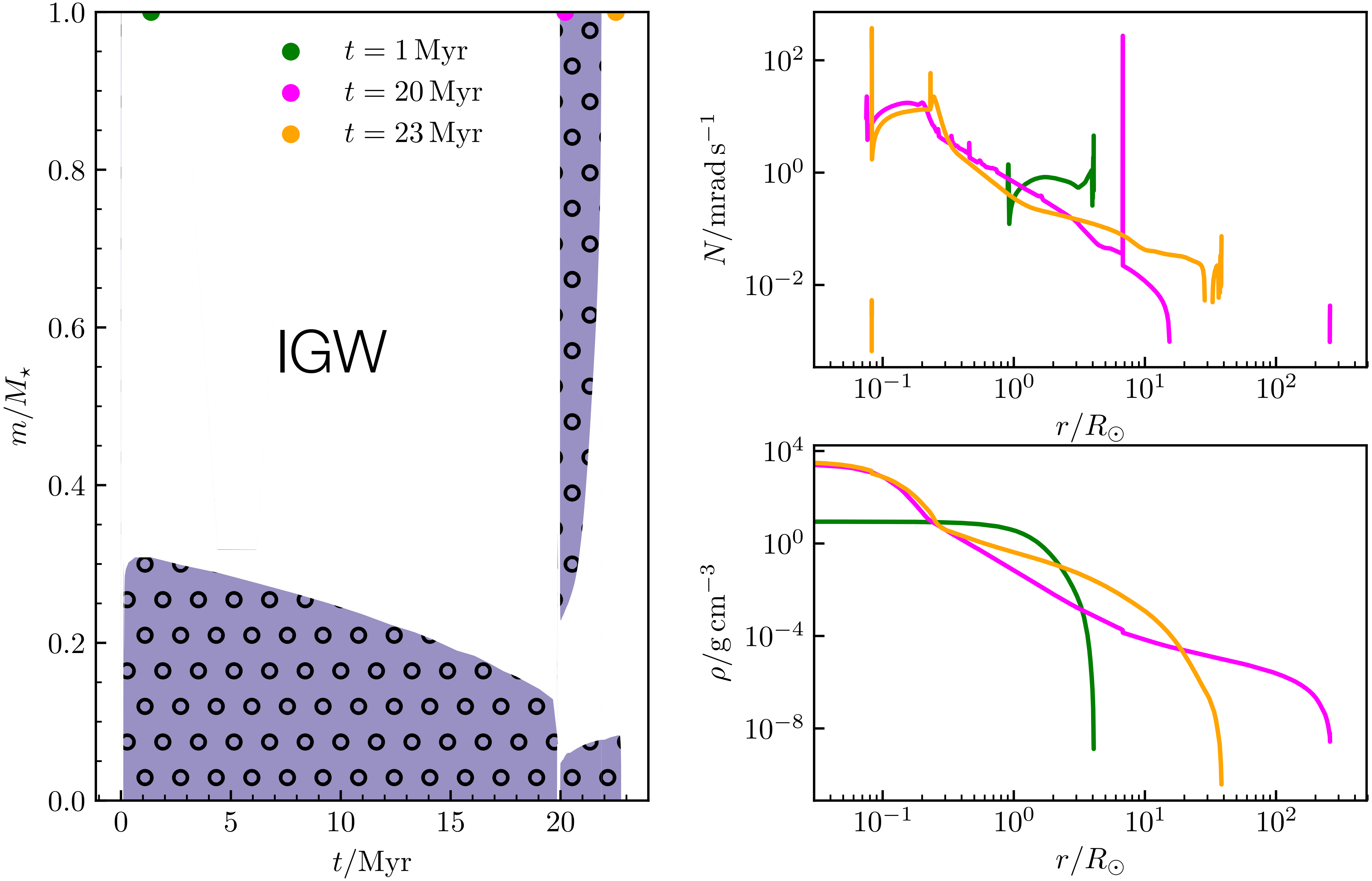
Interior Structure of Stars

$M_{\star,\mathrm{ZAMS}} = 3.0\,\mathrm{M}_{\odot}$

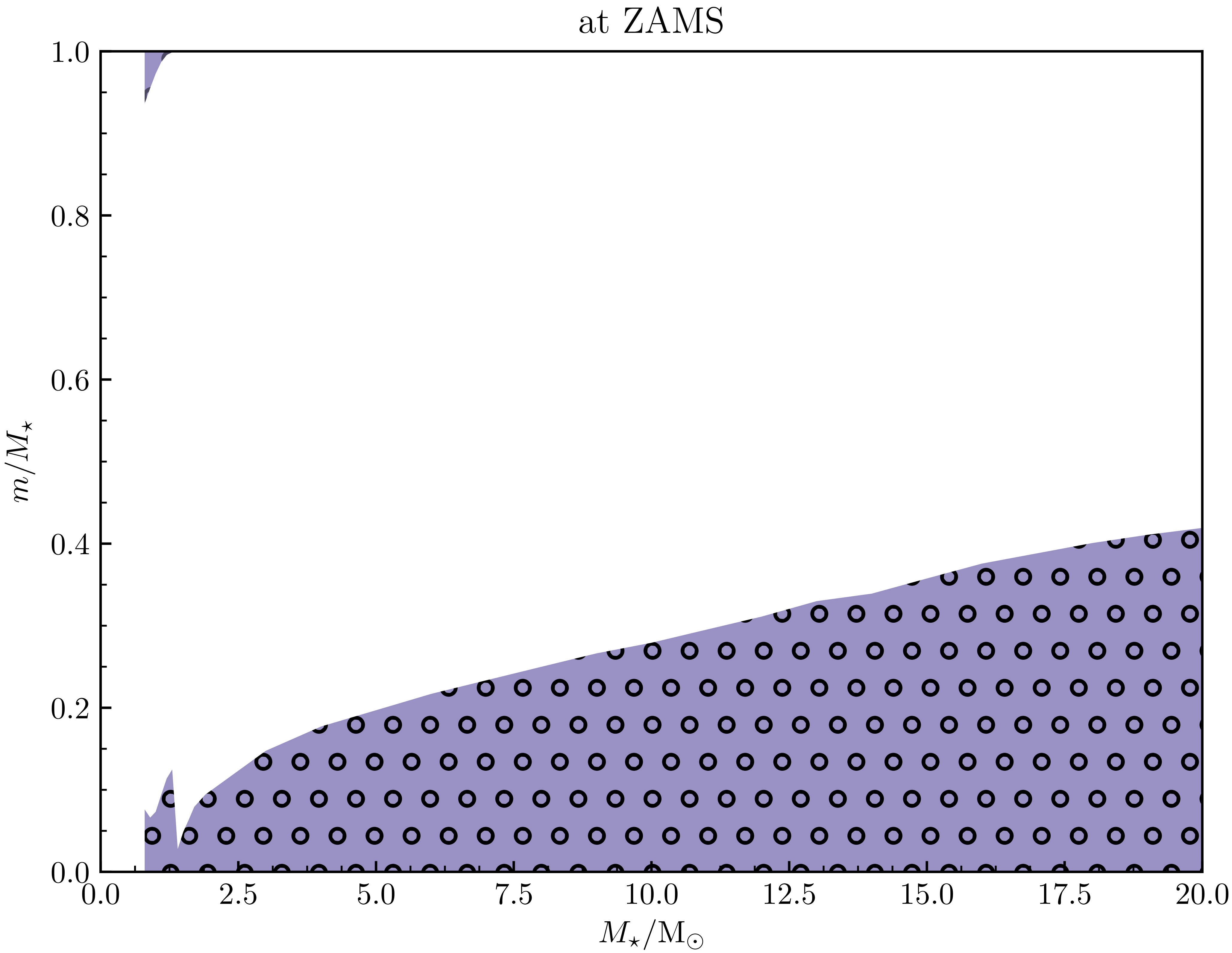


Interior Structure of Stars

$M_{\star,\mathrm{ZAMS}} = 10.0\,\mathrm{M}_\odot$



Interior Structure of Stars



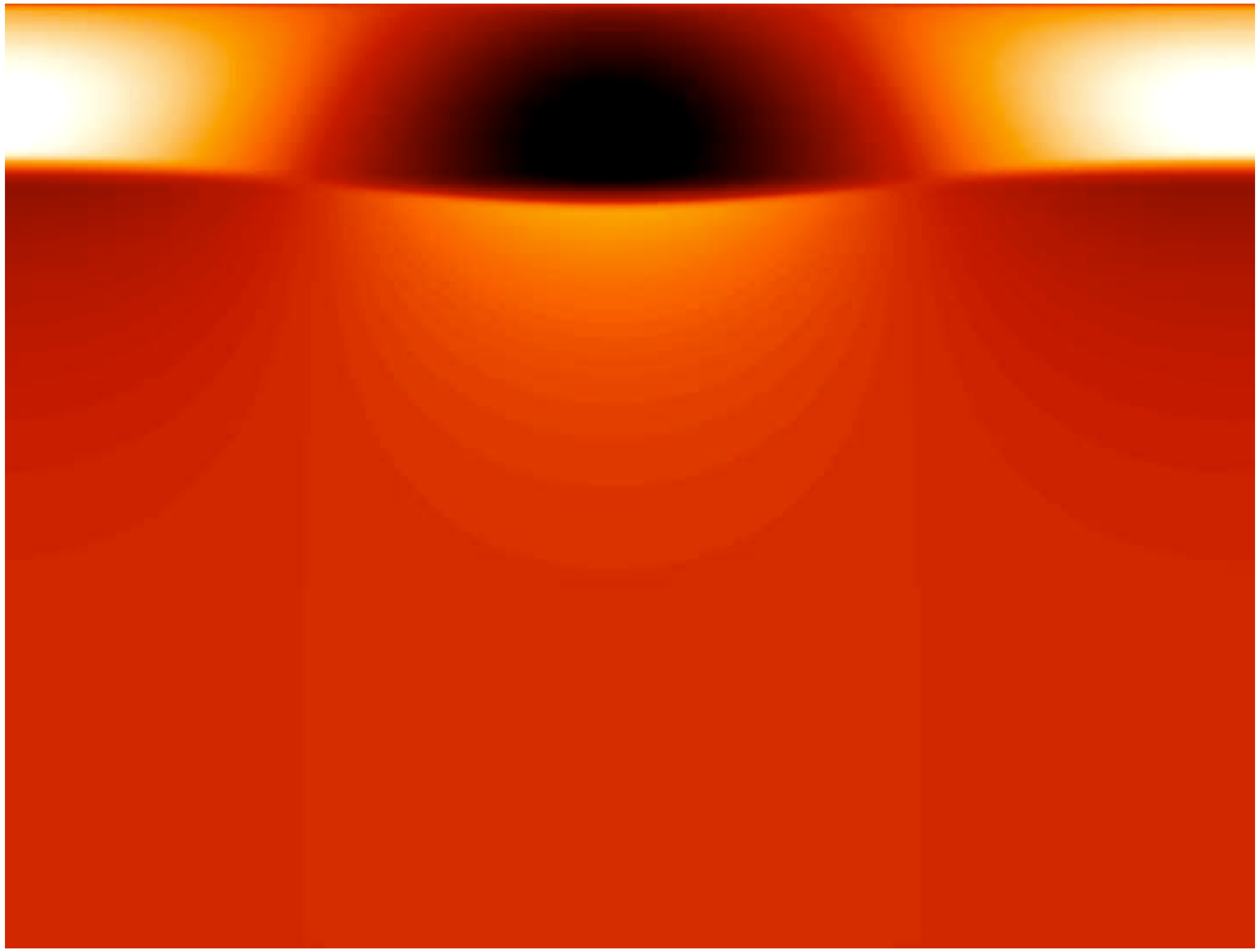
ZAMS=
zero age
main
sequence

Complications to 1D : Convective Overshoot

Mixing Length Theory does not account for non-local mixing either a region is convectively stable or not.

Standard 1D SSE models treat “overshoot” as either exponential or as a step function

But of course it’s likely time and space dependent and its unclear if simple parameterisations are possible/accurate



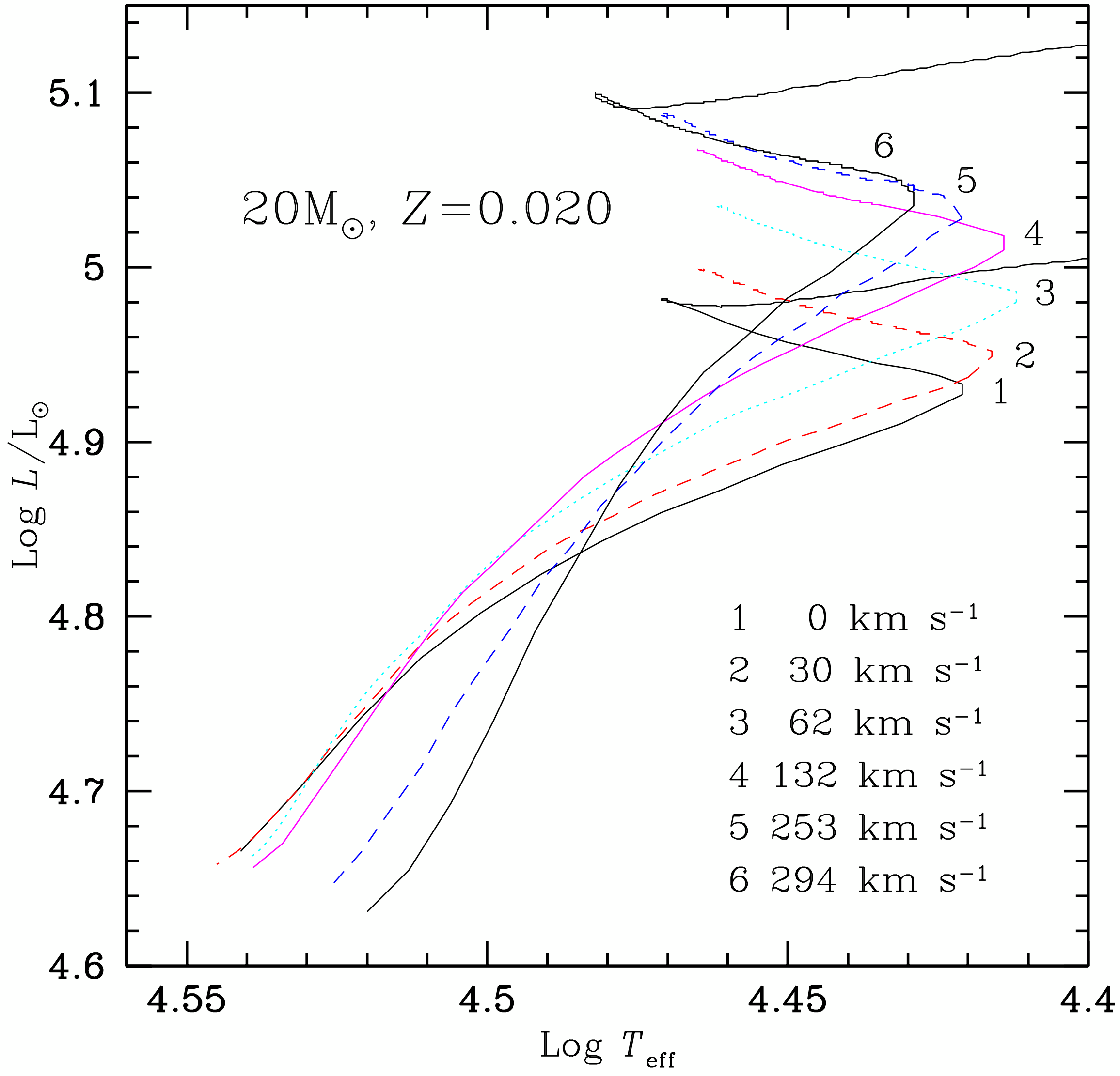
Complications to 1D : Rotation

Rotation has huge impact on stars' life and determines the final fate of massive stars (i.e. explosion and remnant)

Rotation is treated in the *shellular approximation (Zahn 1992)*: rotation is constant on isobars

Causes minor changes to stellar structure equations due to a remap of the system onto isobars rather than just radius

(Generally) Rotation induced mixing (of AM and chemicals) is done via diffusion - in Convection Zone doesn't really matter but in radiation zone makes big difference (next lecture)



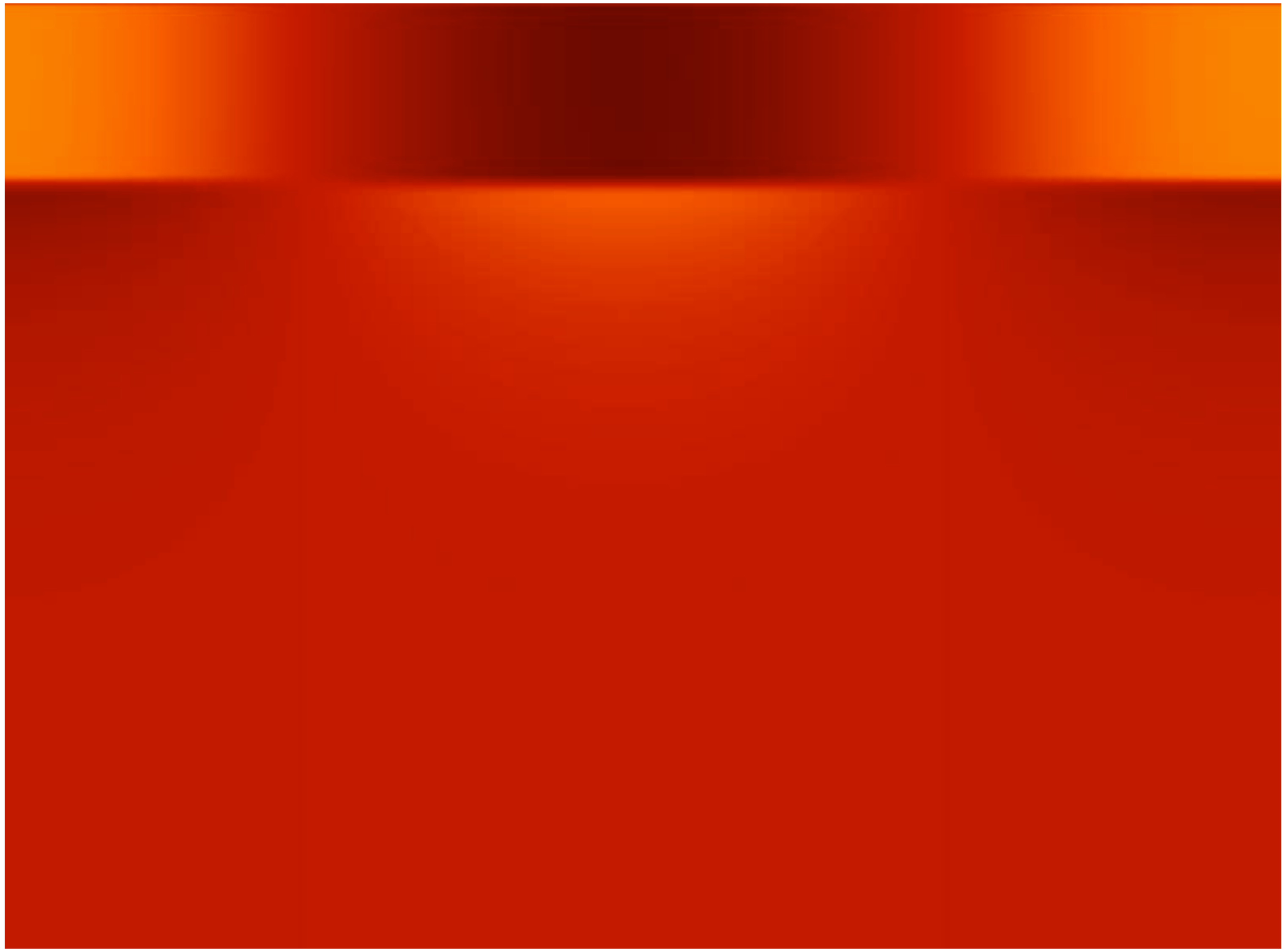
Meynet & Maeder 2000

Complications to 1D : Mixing in Radiative Regions

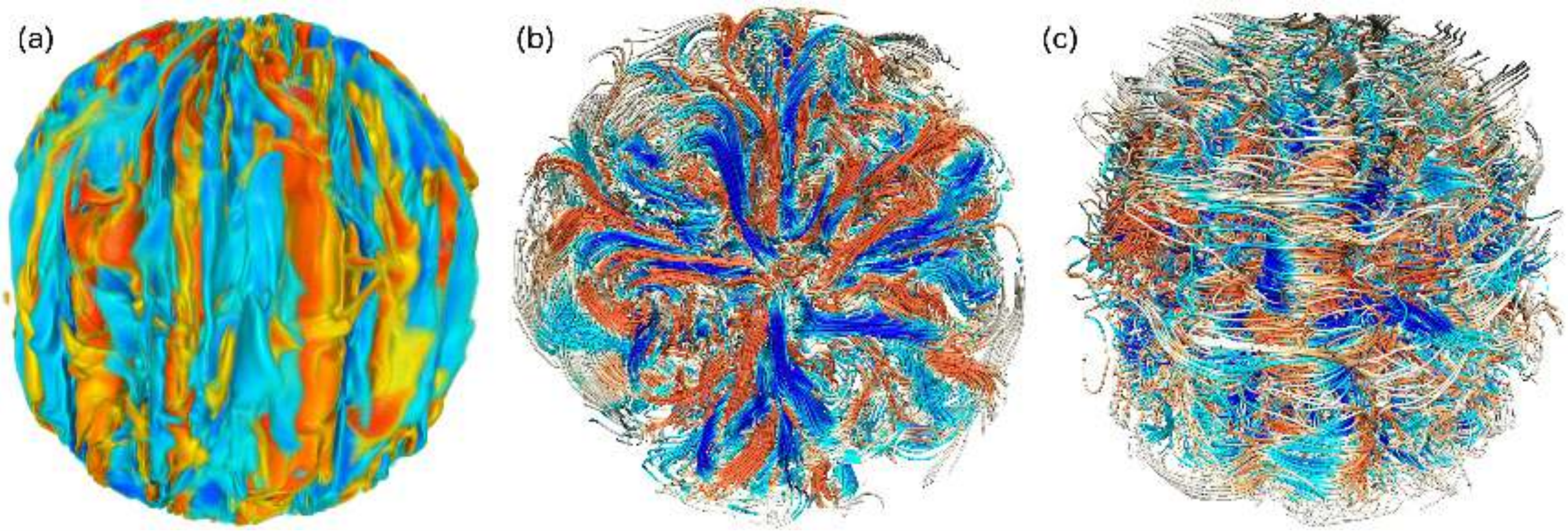
Mixing length theory may be ok in convection zones (as long as we don't need to know too much -i.e didn't work for the Sun's DR) - but not at all appropriate in radiative zones

Host of instabilities (that lead to local turbulence) dynamical/secular shear instability, Solberg-Hoiland instability, Eddington Sweet circulation, GSF are generally not enough to explain observations (next lecture)

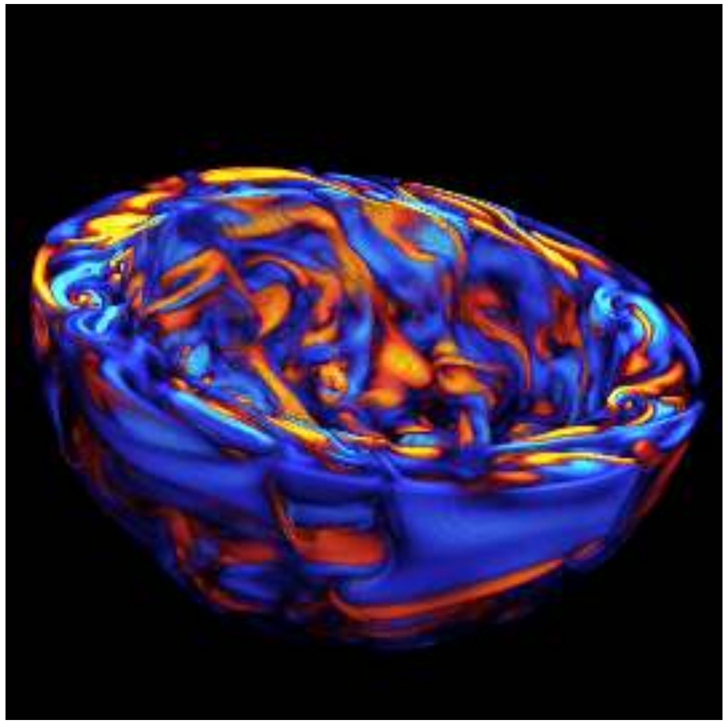
This is where IGW become very important

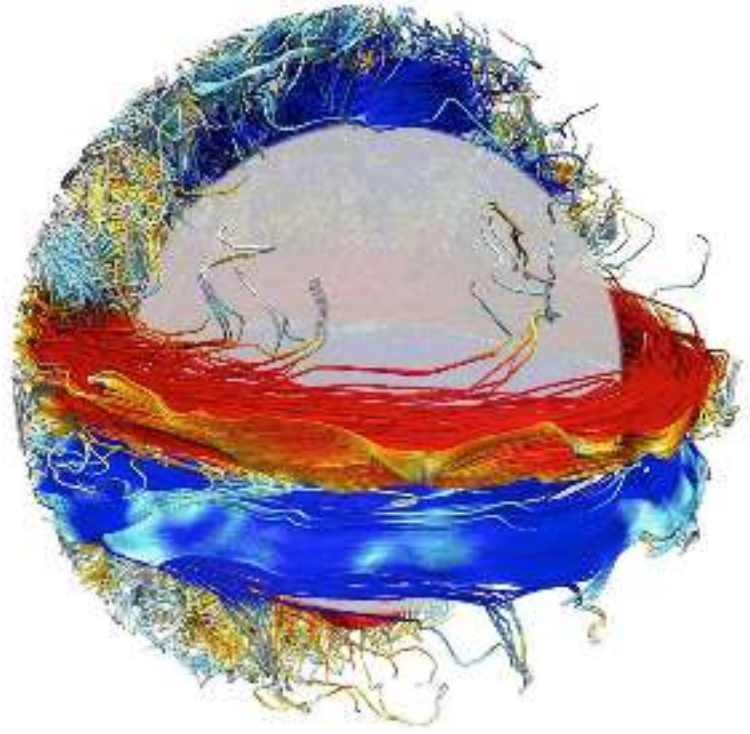


Complications to 1D: Magnetic Fields



We know magnetic fields are important in stars….cant really account for their dynamical effects

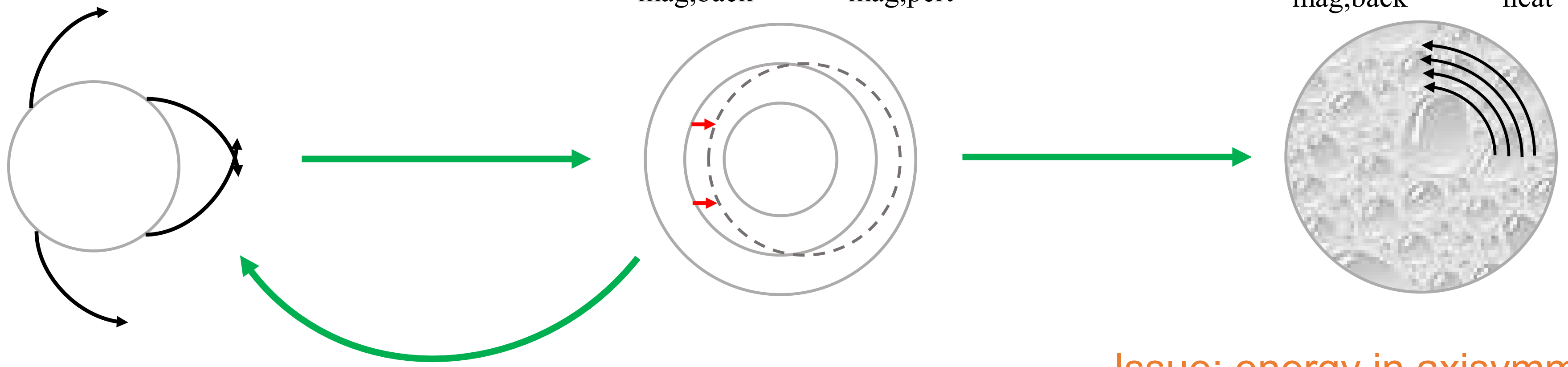




Complications to 1D : Magnetism

Fuller et al. 2019

Tayler-Spruit Dynamo Picture



Winding of δB_r
to generate B_φ
non-axisymmetric
$E_{\rm rot} \rightarrow E_{\rm mag,back}$

Instability generates
δB_r and δv
non-axisymmetric
$E_{\rm mag,back} \rightarrow E_{\rm mag,pert}$

Turbulent dissipation of
background magnetic energy
non-axisymmetric
$E_{\rm mag,back} \rightarrow E_{\rm heat}$

Issue: no axisymmetric B_φ created
by winding non-axisymmetric δB_r

Issue: energy in axisymmetric
B_φ not rapidly dissipated

Maxwell Stress

$$S_m = \frac{B_r B_\phi}{4\pi} = \rho \nu_e r \frac{\partial \Omega}{\partial r} = \rho r^2 \Omega^2 q^3 \left(\frac{\Omega}{N} \right)^4$$

$$q = \frac{r}{\Omega} \frac{\partial \Omega}{\partial r}$$

(Modern version with different saturation Fuller et al. 2019)