Part III:Parametric instabilities and turbulence.

Lecture by J. Noir at WITGAF Cargese, Corsica, 2019

1. A primer on Inviscid Parametric Instabilities.
2. Threshold and dynamics near onset.
3. Geostrophic .vs. waves turbulence.

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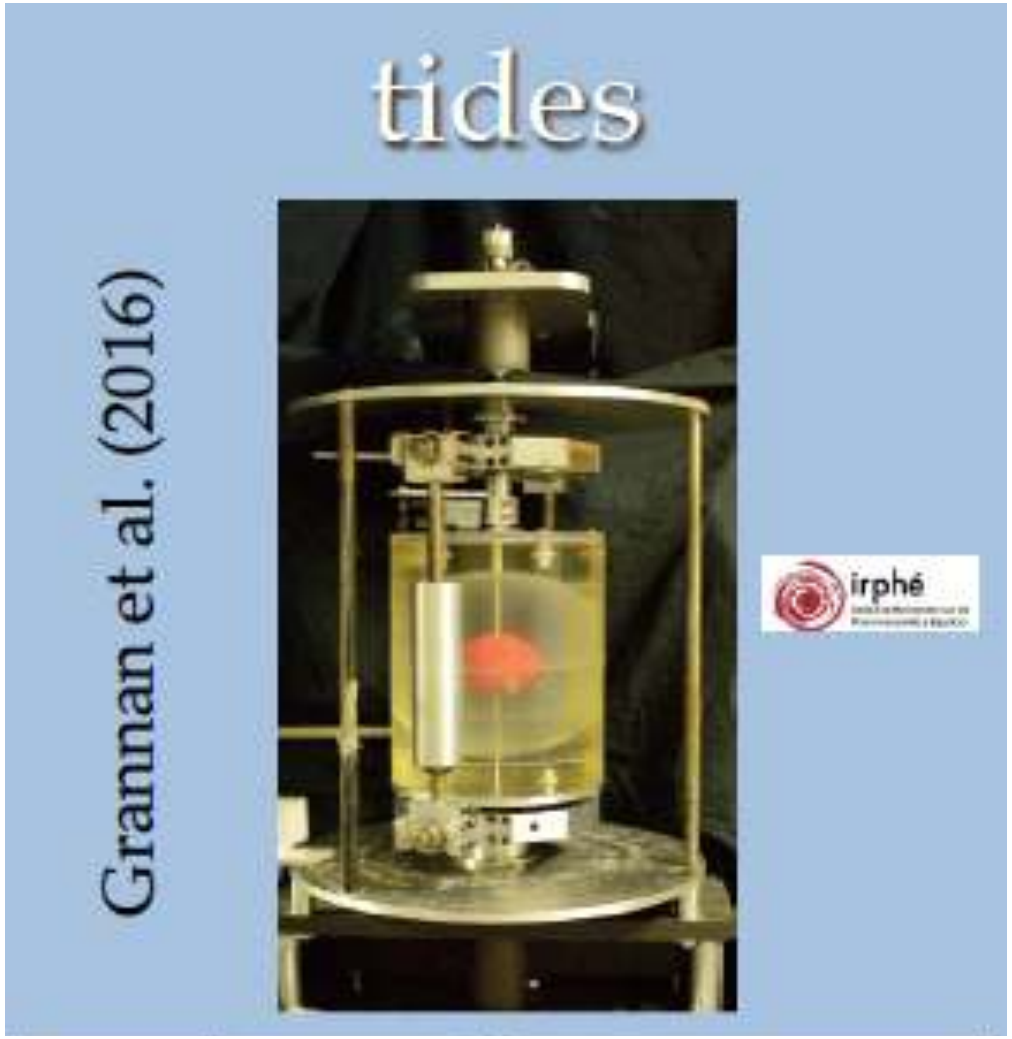
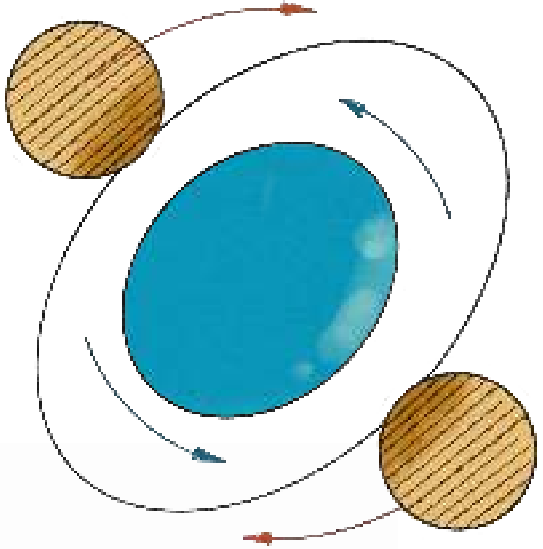
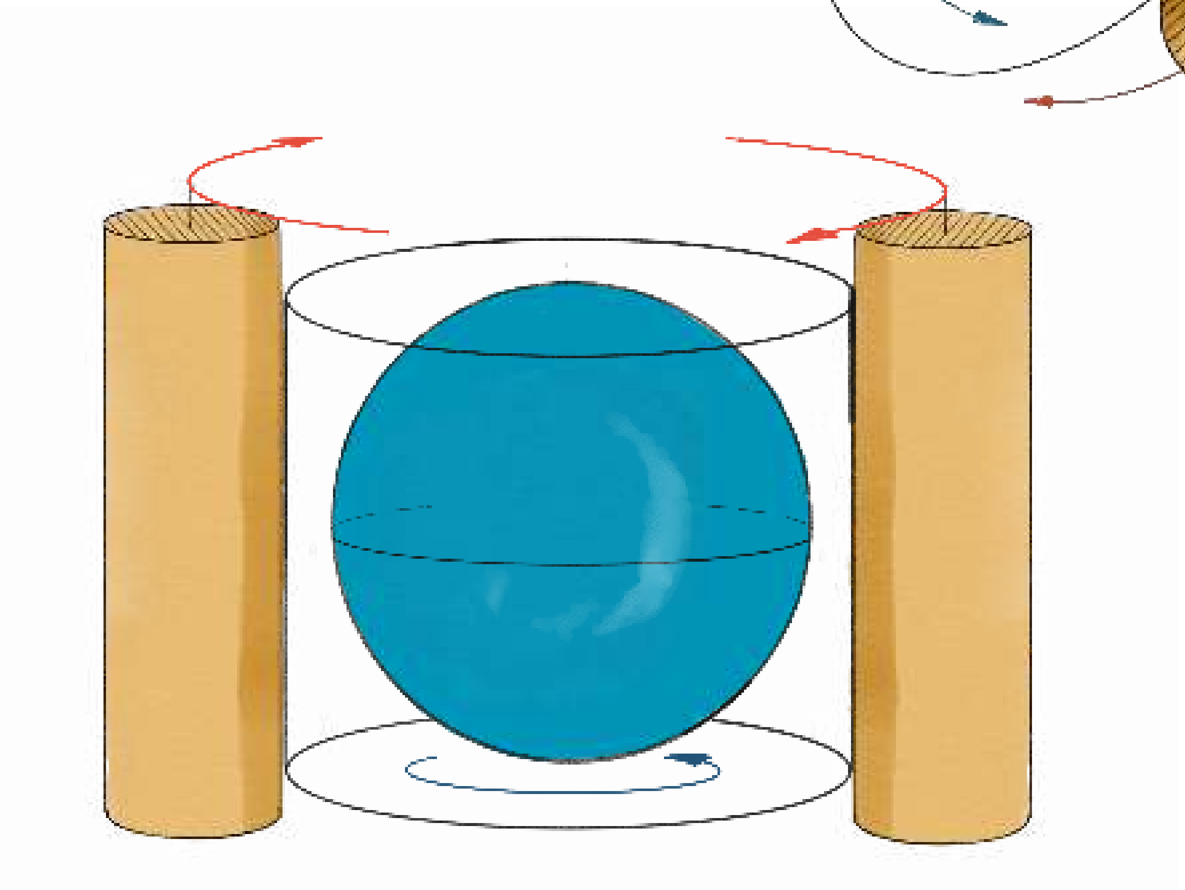
base flow

$\vec{u}_b, \ \omega_b$



This setup mimic the forcing produced by a travelling tide.





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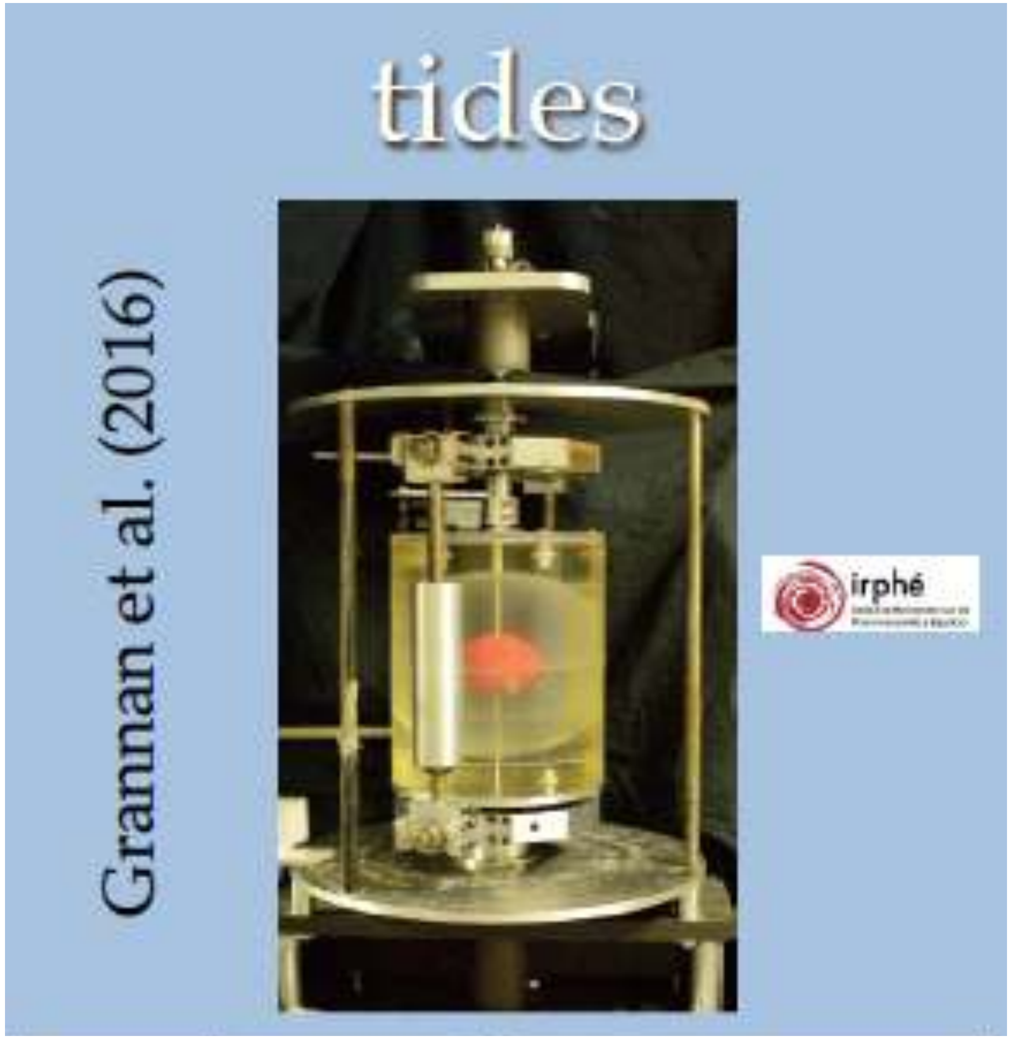
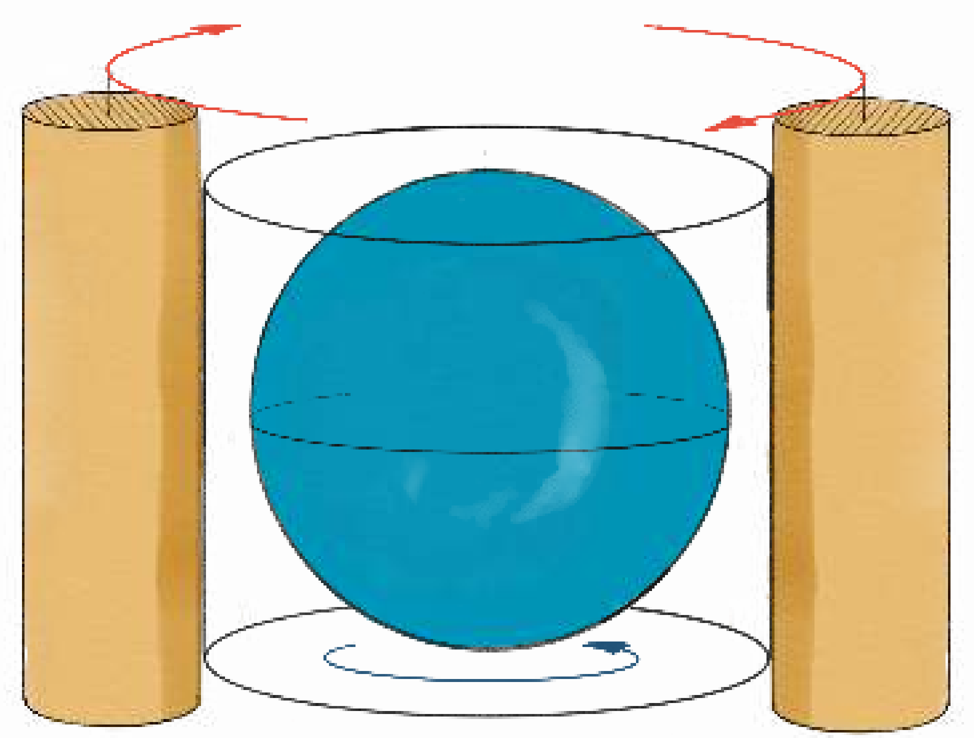
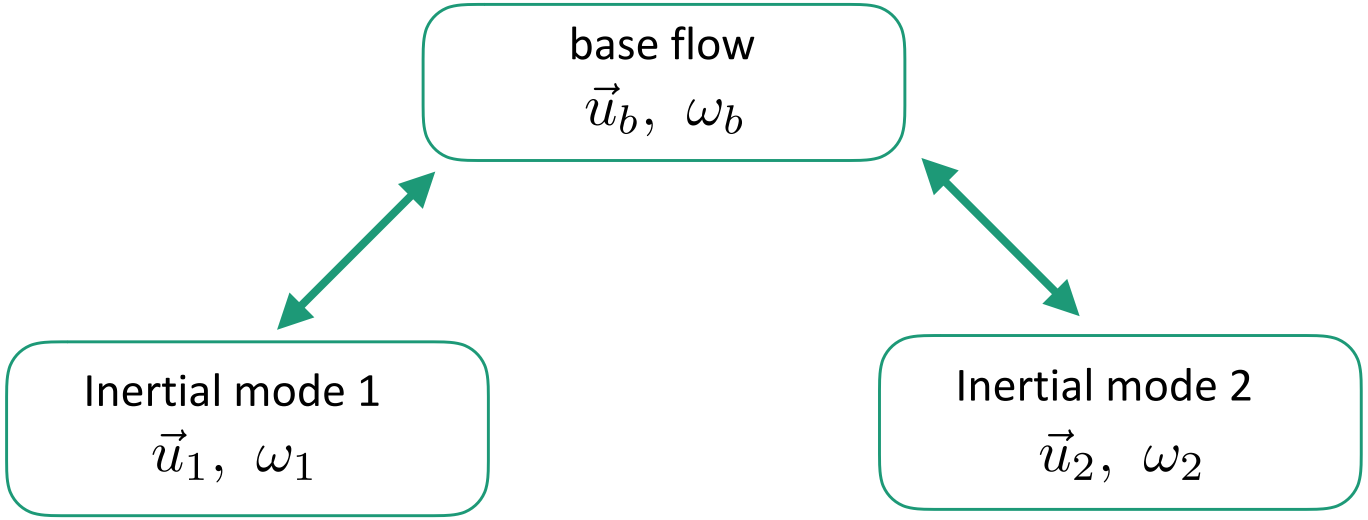


$$\vec{u}_b = \vec{\Omega} \times \vec{r} + \vec{\nabla}\phi(\eta, t)$$





$$\vec{u}_b = \vec{\Omega} \times \vec{r} + \vec{\nabla}\phi(\eta, t)$$

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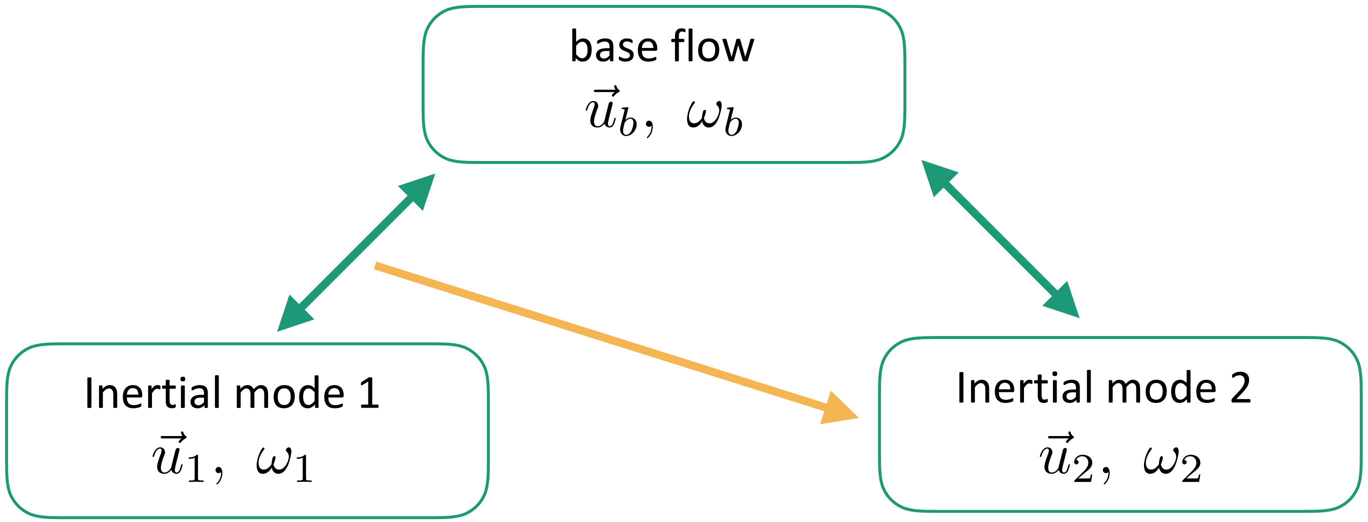

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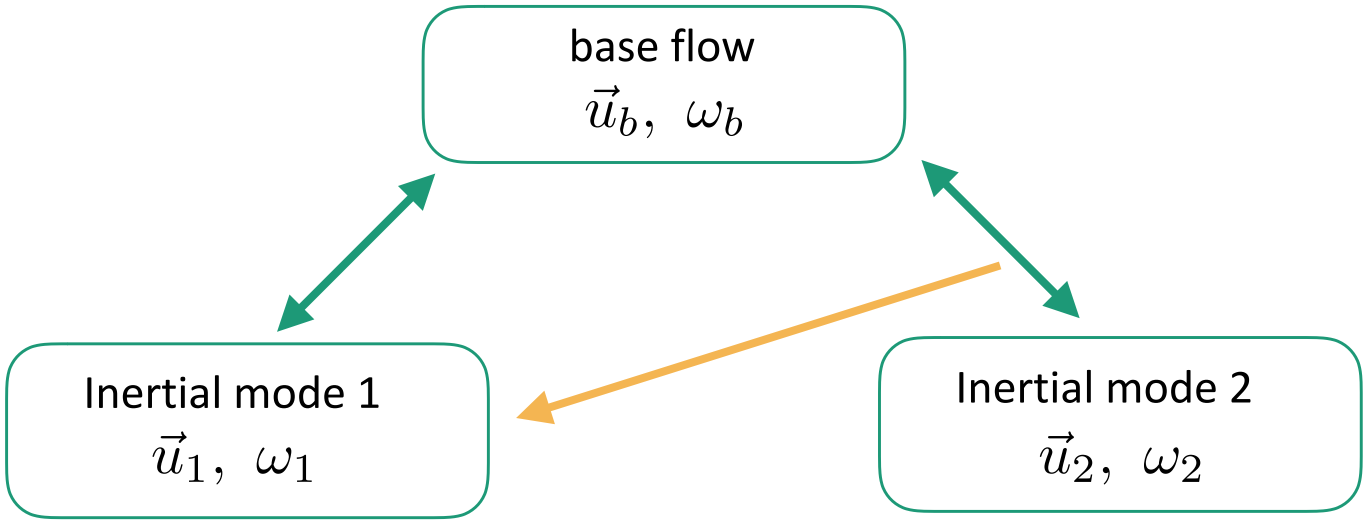

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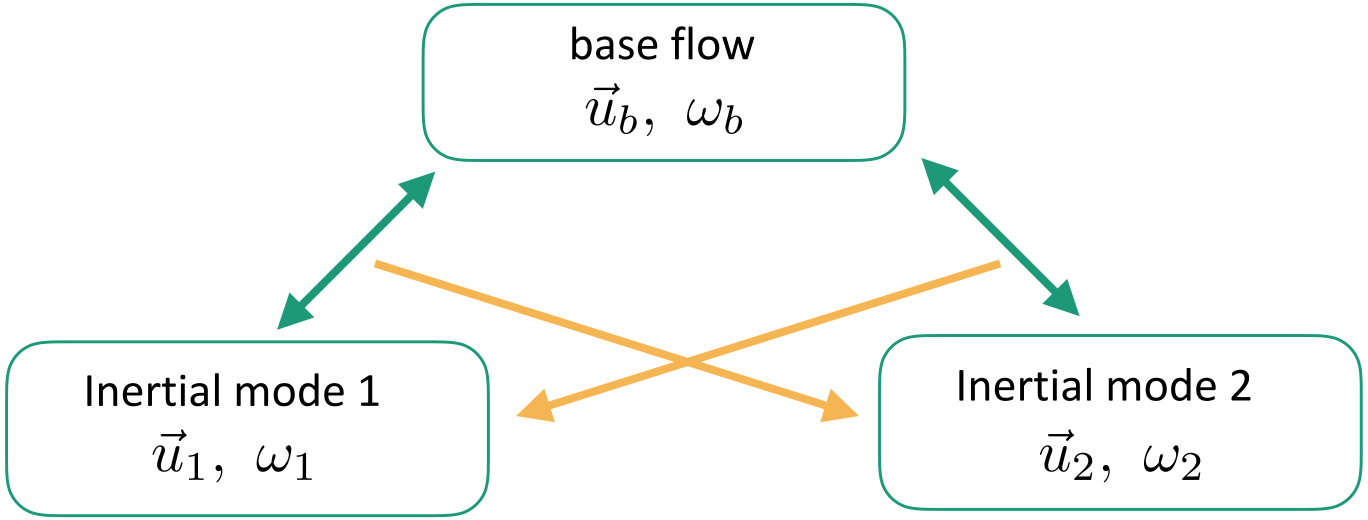

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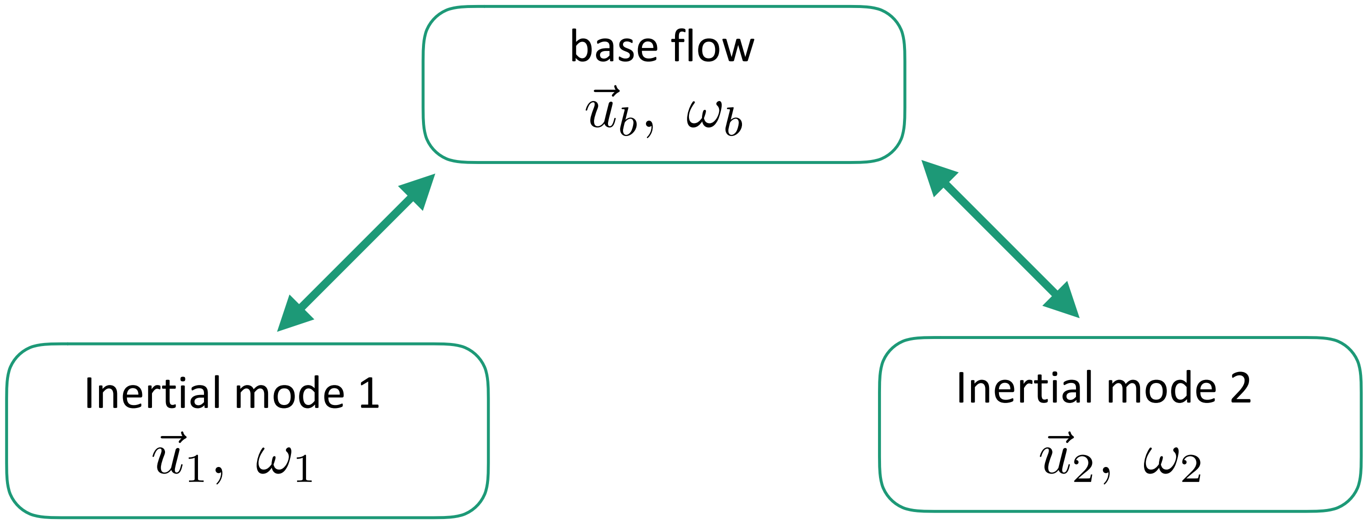

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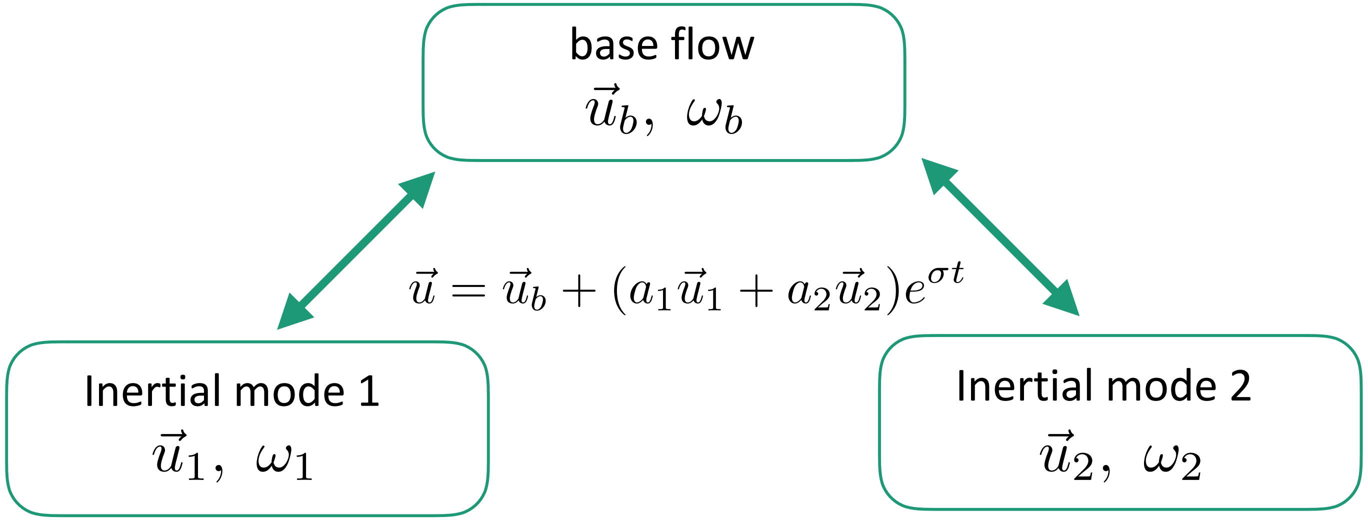

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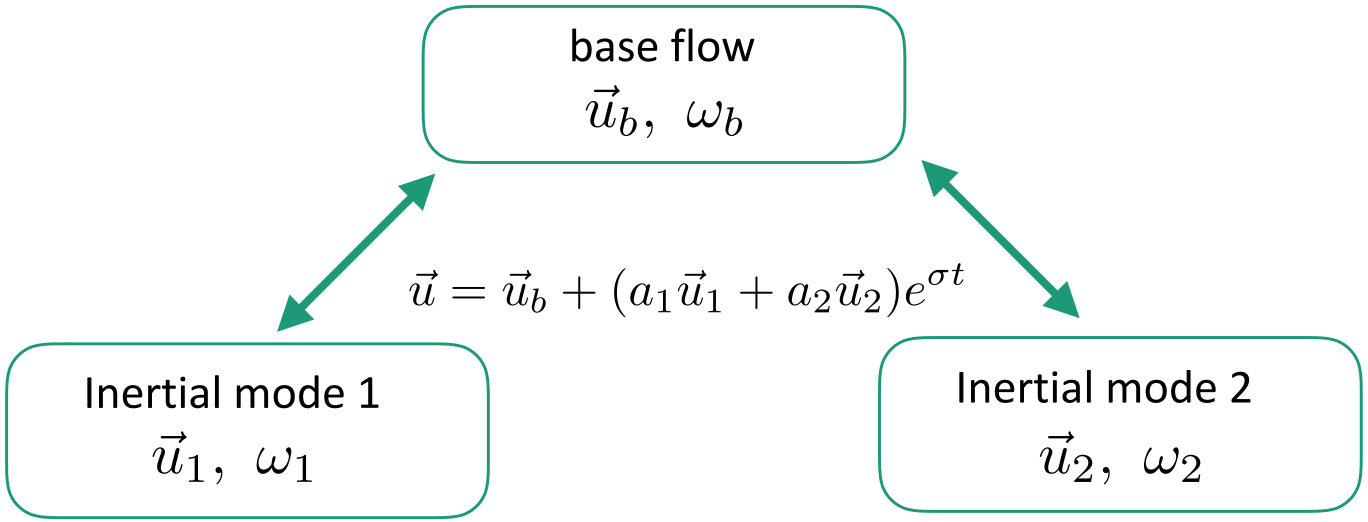
At the onset of the instability we assume an exponential growth, near onset we get: $(a_1, a_2)e^{\sigma t} \propto \varepsilon << 1$



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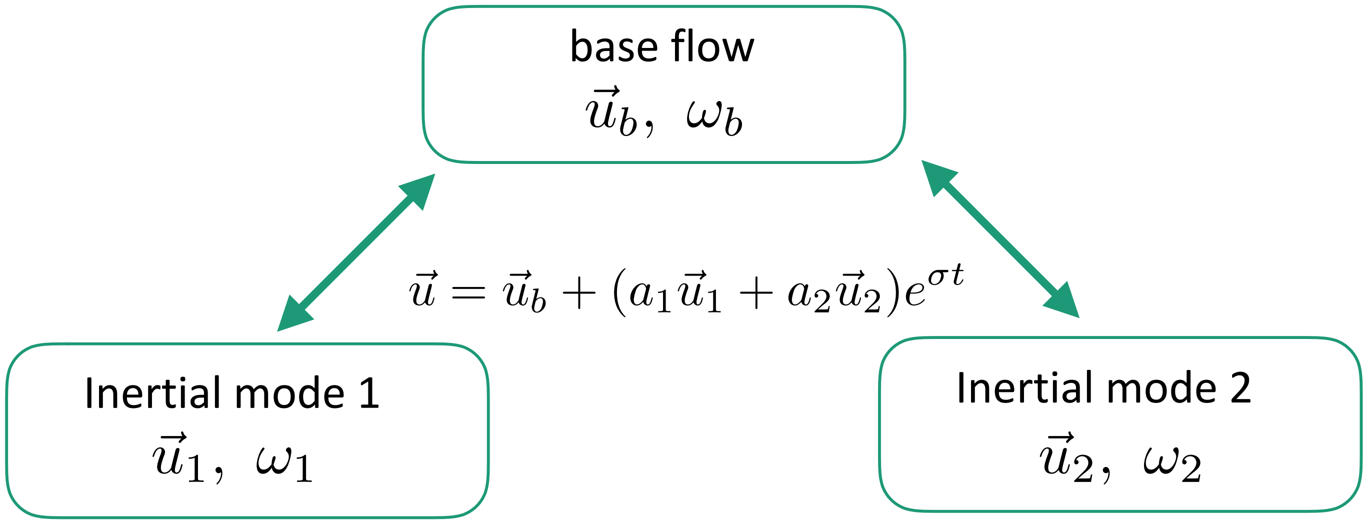

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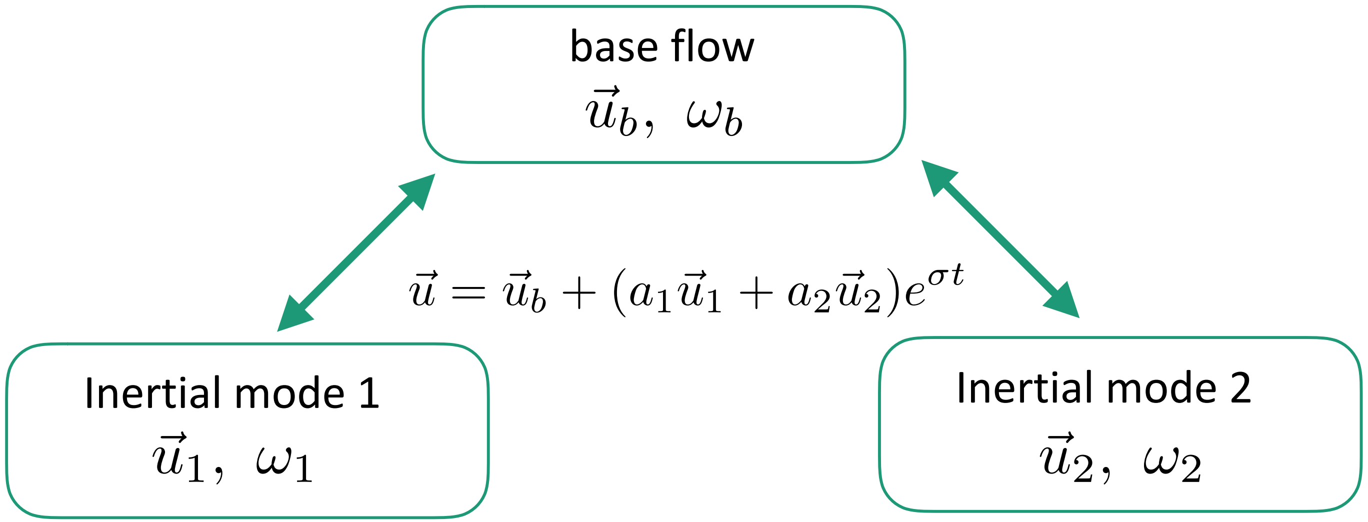

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$0, 2\omega_1$ $\qquad$ $\omega_b, \omega_b \pm 2\omega_1$ $\qquad$ $\omega_b, \omega_b \pm 2\omega_1$

$$\sigma a_2 \langle \vec{u}_2 \cdot \vec{u}_2 \rangle = -\langle \vec{u}_2 \cdot (\vec{u}_b \cdot \nabla(a_1\vec{u}_1 + a_2\vec{u}_2))\rangle - \langle \vec{u}_2 \cdot ((a_1\vec{u}_1 + a_2\vec{u}_2) \cdot \nabla \vec{u}_b)\rangle$$

$0, 2\omega_2$ $\qquad$ $\omega_b, \omega_b \pm 2\omega_2$ $\qquad$ $\omega_b, \omega_b \pm 2\omega_2$

$$\omega_1 = \omega_2 = \pm\frac{\omega_b}{2}$$

The modes 1 and 2 are identical, one traveling in a prograde direction, the other one in the retrograde direction.

$$\sigma a_1 \langle \vec{u}_1 \cdot \vec{u}_1 \rangle = -\langle \vec{u}_1 \cdot (\vec{u}_b \cdot \nabla(a_1\vec{u}_1 + a_2\vec{u}_2)) \rangle - \langle \vec{u}_1 \cdot ((a_1\vec{u}_1 + a_2\vec{u}_2) \cdot \nabla\vec{u}_b) \rangle$$

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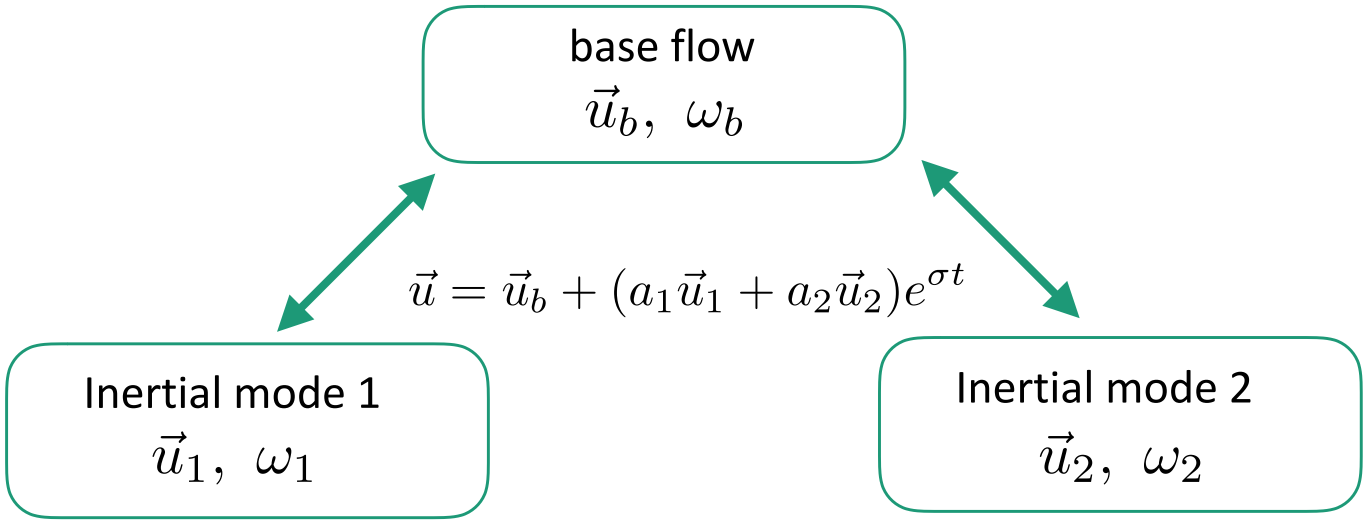

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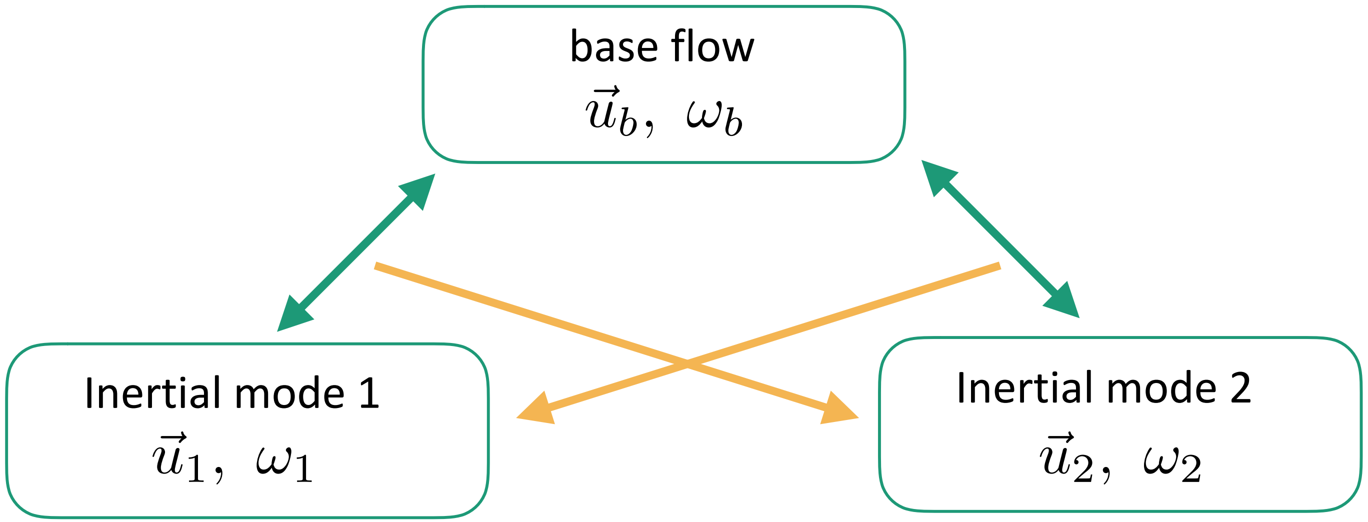
$$\sigma a_1 \langle \vec{u}_1 \cdot \vec{u}_1 \rangle = -a_2 \left[\langle \vec{u}_1 \cdot \vec{u}_b \cdot \nabla \vec{u}_2 \rangle + \langle \vec{u}_1 \cdot \vec{u}_2 \cdot \nabla \vec{u}_b \rangle \right] \qquad \omega_2 = \omega_b \pm \omega_1$$

$$\sigma a_2 \langle \vec{u}_2 \cdot \vec{u}_2 \rangle = -a_1 \left[\langle \vec{u}_2 \cdot \vec{u}_b \cdot \nabla \vec{u}_1 \rangle + \langle \vec{u}_2 \cdot \vec{u}_1 \cdot \nabla \vec{u}_b \rangle \right]$$



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$$C_1 = \frac{\langle \vec{u}_1 \cdot \vec{u}_b \cdot \nabla \vec{u}_2 \rangle + \langle \vec{u}_1 \cdot \vec{u}_2 \cdot \nabla \vec{u}_b \rangle}{\langle \vec{u}_1 \cdot \vec{u}_1 \rangle}$$

$$\sigma = \sqrt{C_1 C_2}$$

$$C_2 = \frac{\langle \vec{u}_2 \cdot \vec{u}_b \cdot \nabla \vec{u}_1 \rangle + \langle \vec{u}_2 \cdot \vec{u}_1 \cdot \nabla \vec{u}_b \rangle}{\langle \vec{u}_2 \cdot \vec{u}_2 \rangle}$$

$$\omega_2 = \omega_b \pm \omega_1$$

In spheroidal shell and annulus, the separation of variable in azimuth, latitude / vertical component, leads to additional parametric resonant conditions when carrying the volume integral.

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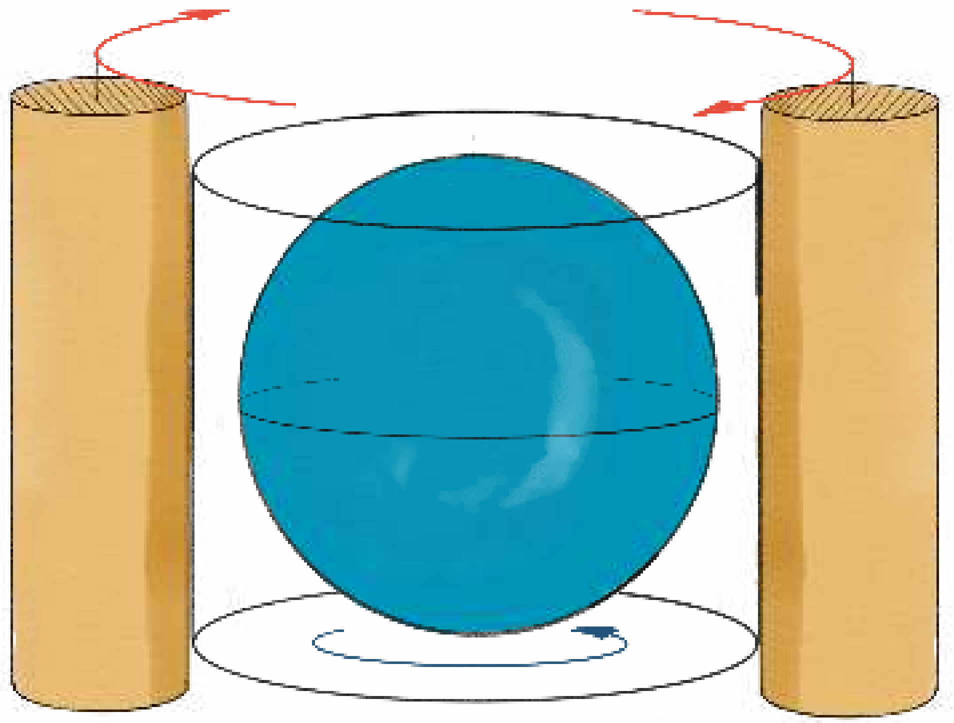
For small enough departure of the cavity from a sphere, the inertial modes contributing to the parametric instability are those of a pure sphere. Using a spherical coordinate system, one can separate the azimuthal coordinate from the latitudinal and radial coordinates.



$$\vec{u}_j = \vec{U}_j(r,\theta)e^{im\Phi}$$

$$C_1 = \frac{\langle \vec{u}_1 \cdot \vec{u}_b \cdot \nabla \vec{u}_2 \rangle + \langle \vec{u}_1 \cdot \vec{u}_2 \cdot \nabla \vec{u}_b \rangle}{\langle \vec{u}_1 \cdot \vec{u}_1 \rangle}$$

$$C_1 \propto \int_0^{2\pi} e^{i[m_b \pm m_1 \pm m_2]\phi} d\Phi \qquad m_2 = m_b \pm m_1$$

The rate at which the instability grow results from the balance of the rate at which energy is deposited on the modes (σ) and the rate at which these modes dissipate the energy by viscous friction in the boundary layer ~ $\mathcal{K}\sqrt{E}$

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$$i\omega_2 \vec{u}_1 + 2\vec{\Omega} \times \vec{u}_2 = -\vec{\nabla} p_2$$

$$< \vec{u}_1 \cdot \vec{u}_2 >= \int_V \vec{u}_1 \cdot \vec{u}_2^{\dagger} dV = 0$$

The rate at which the instability grow results from the balance of the rate at which energy is deposited on the modes (σ) and the rate at which these modes dissipate the energy by viscous friction in the boundary layer ~ $\mathcal{K}\sqrt{E}$

So, for this type of instabilities to develop a minimum of conditions must be satisfied:

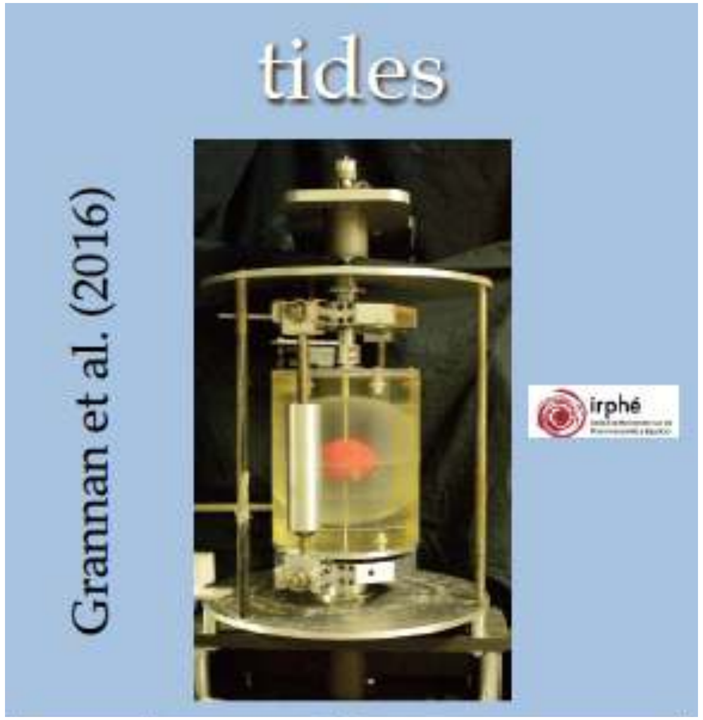
$$\omega_2 = \omega_b \pm \omega_1$$

$$\sqrt{C_1 C_2} - \mathcal{K}\sqrt{E} > 0$$

$$\omega_2 = \omega_b \pm \omega_1$$
$$m_2 = m_b \pm m_1$$

Each cavity shape will lead to additional geometrical conditions on the mode, in spheroidal, spherical, cylindrical shells a strict condition is:

$$m_2 = m_b \pm m_1$$



Precession driven elliptical instability

$$\frac{|\Omega_m - \Omega_c|}{\Omega_c} \gtrsim \eta E^{1/2}$$

$$\frac{|\Omega_m - \Omega_c|}{\Omega_c} \gtrsim \eta^{1/2} E^{1/4}$$

Kerswell 1993

Precession conical shear instability

$$\frac{|\Omega_m - \Omega_c|}{\Omega_c} \gtrsim E^{3/10}$$

Goto et al. 2014, Lin et al. 2015

libration driven elliptical instability

$$\sigma = \frac{16 + f^2}{64} \beta \varepsilon - K E^{1/2}$$

f libration frequency

ε libration amplitude

tides driven elliptical instability

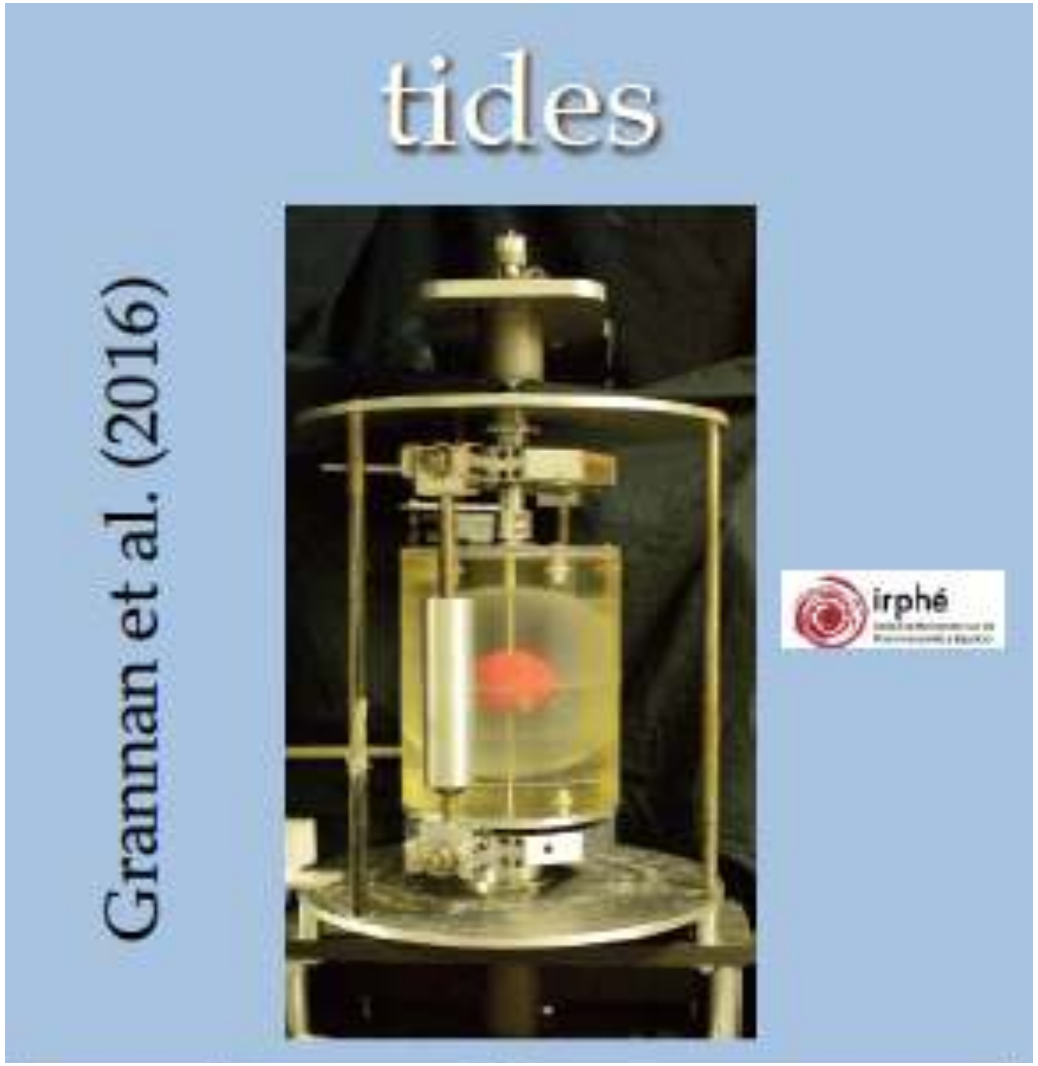
$$\sigma = \frac{(3-\Omega)^2}{16} |1-\Omega| \beta - K E^{1/2}$$

$\Omega = \Omega_{orb} / \Omega_{spin}$

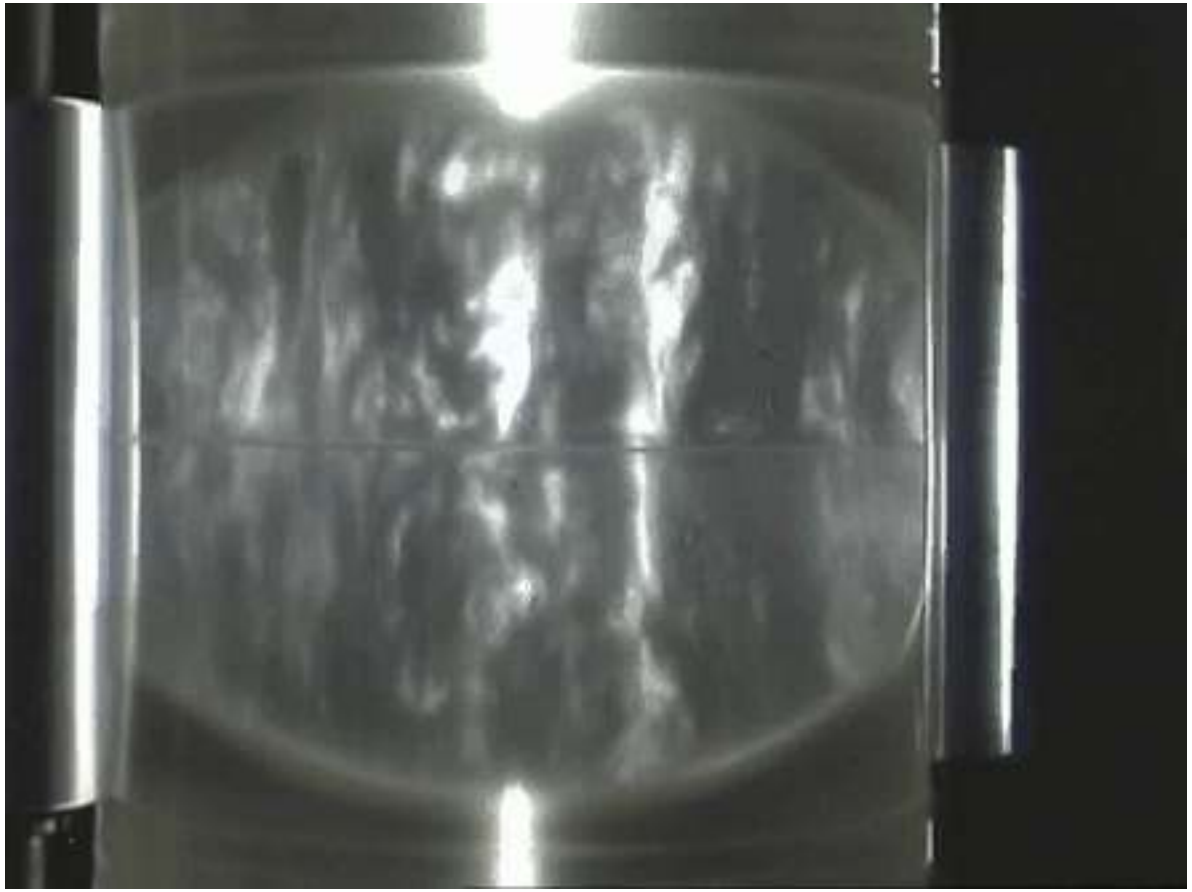
Le Bars et al. 2011

1. A primer on Inviscid Parametric Instabilities.
2. Threshold and dynamics near onset.
3. Geostrophic .vs. waves turbulence.

Except in a very narrow range of control parameters near onset, the parametric instability itself breaks down leading to space filling turbulence during the collapse phase usually followed by a growth phase and so on. The growth and collapse can be quasi periodic, aperiodic or chaotic. In some cases, after the initial growth no clear growth and collapse are observed, the flow remains turbulent.



Tidally driven elliptical instability (TDEI)



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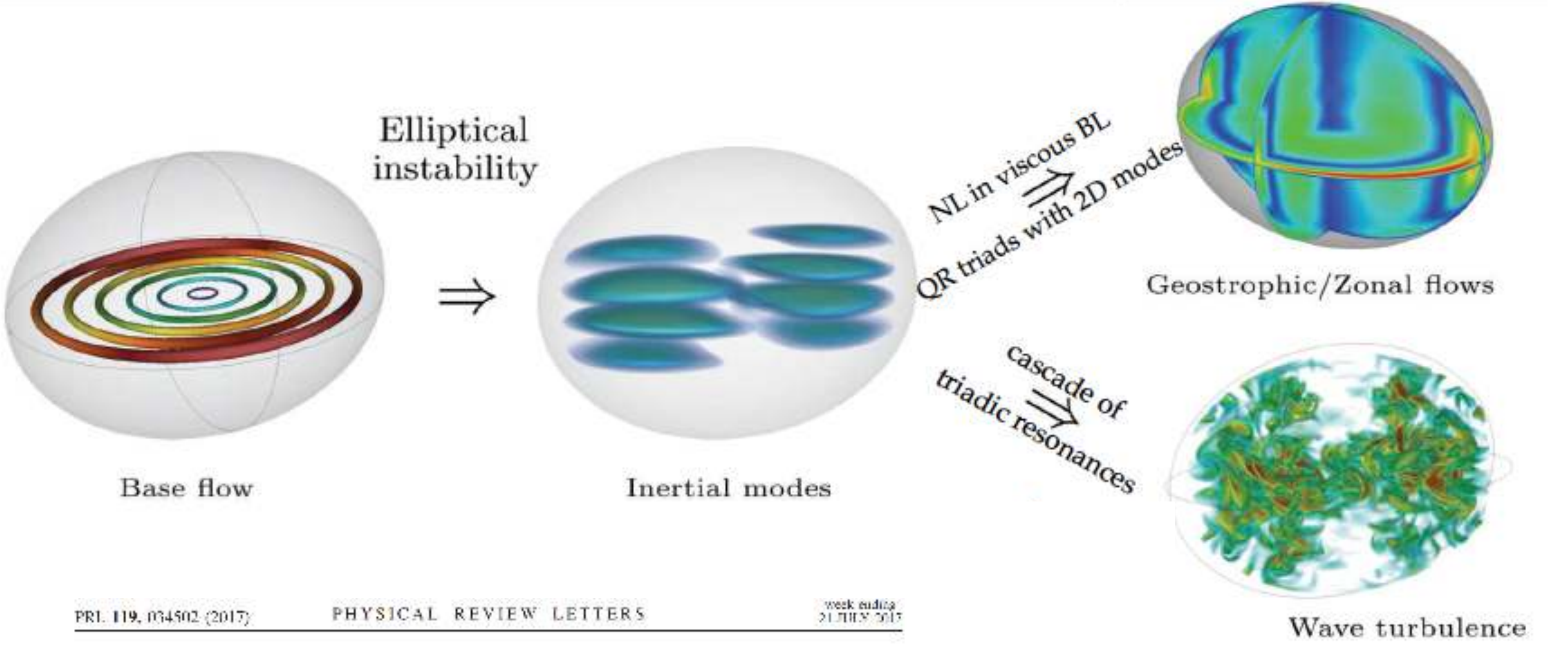
The nonlinear fate of the elliptical instability



PRL 119, 034502 (2017) PHYSICAL REVIEW LETTERS week ending 21 JULY 2017

Inertial Wave Turbulence Driven by Elliptical Instability

Thomas Le Reun,[1] Benjamin Favier,[1] Adrian J. Barker,[2] and Michael Le Bars[1]
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[2]Department of Applied Mathematics, School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom
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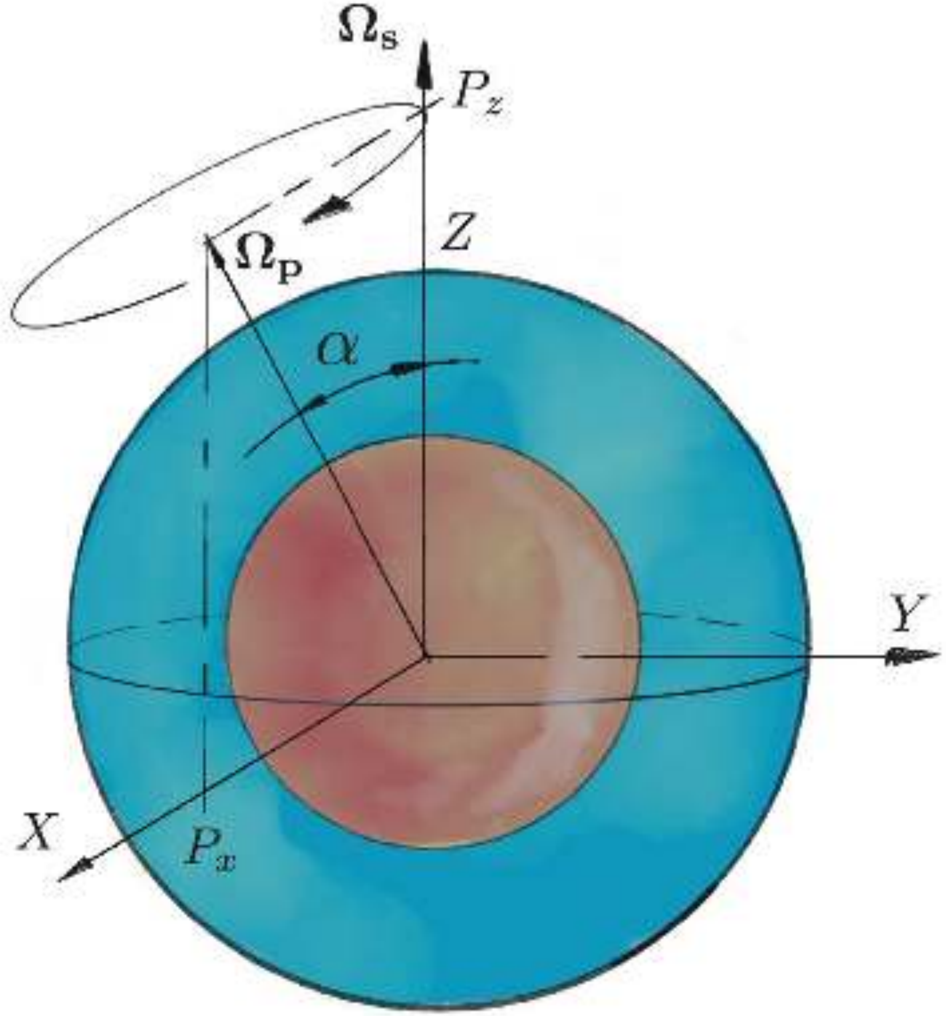

Y. Lin, P. Marti and J. Noir

„Shear-driven parametric instability in a precessing sphere", POF, 2015

D. Cébron, R. Laguerre, J. Noir and N. Schaeffer.

"Precessing spherical shells : flows , dissipation , dynamo and the lunar core", GJI, 2019

For large enough differential rotation between the liquid and the solid shells, boundary layers and bulk instabilities develop in the system:

$$Po = 8 \times 10^{-3},\ E = 3 \times 10^{-5},\ \alpha = 120^{\circ}$$

a) b) c)

Isosurfaces of the anti-symmetric energy in the mean fluid rotating frame frame for (a) $\eta = 0.01$, $Po = 8 \times 10^{-3}$ (b) $\eta = 0.3$, $Po = 8 \times 10^{-3}$ and (c) $\eta = 0.7$, $Po = 8.5 \times 10^{-3}$. In each case $\alpha = 120°$ and $E = 3.0 \times 10^{-5}$. Color correspond to positive (black) and negative (white) axial velocity. The snapshots are taken during the initial growth phase of the instability.

D. Cébron, R. Laguerre, J. Noir and N. Schaeffer.

“Precessing spherical shells : flows , dissipation , dynamo and the lunar core”, GJI, 2019

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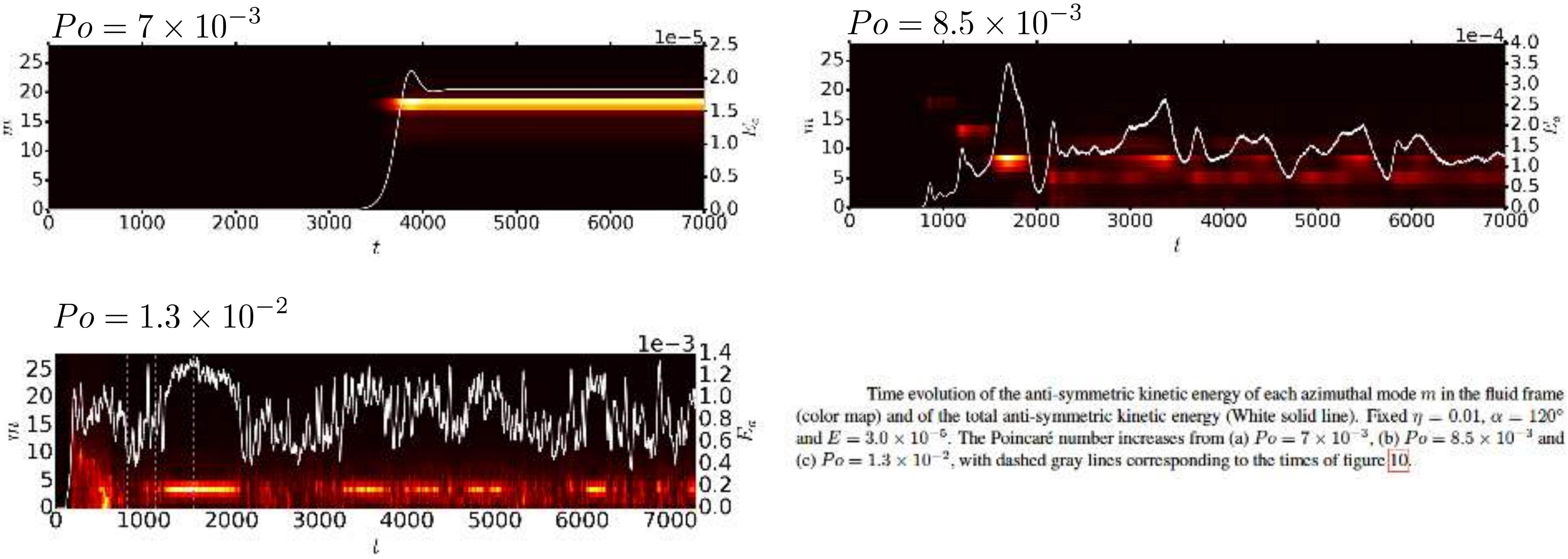


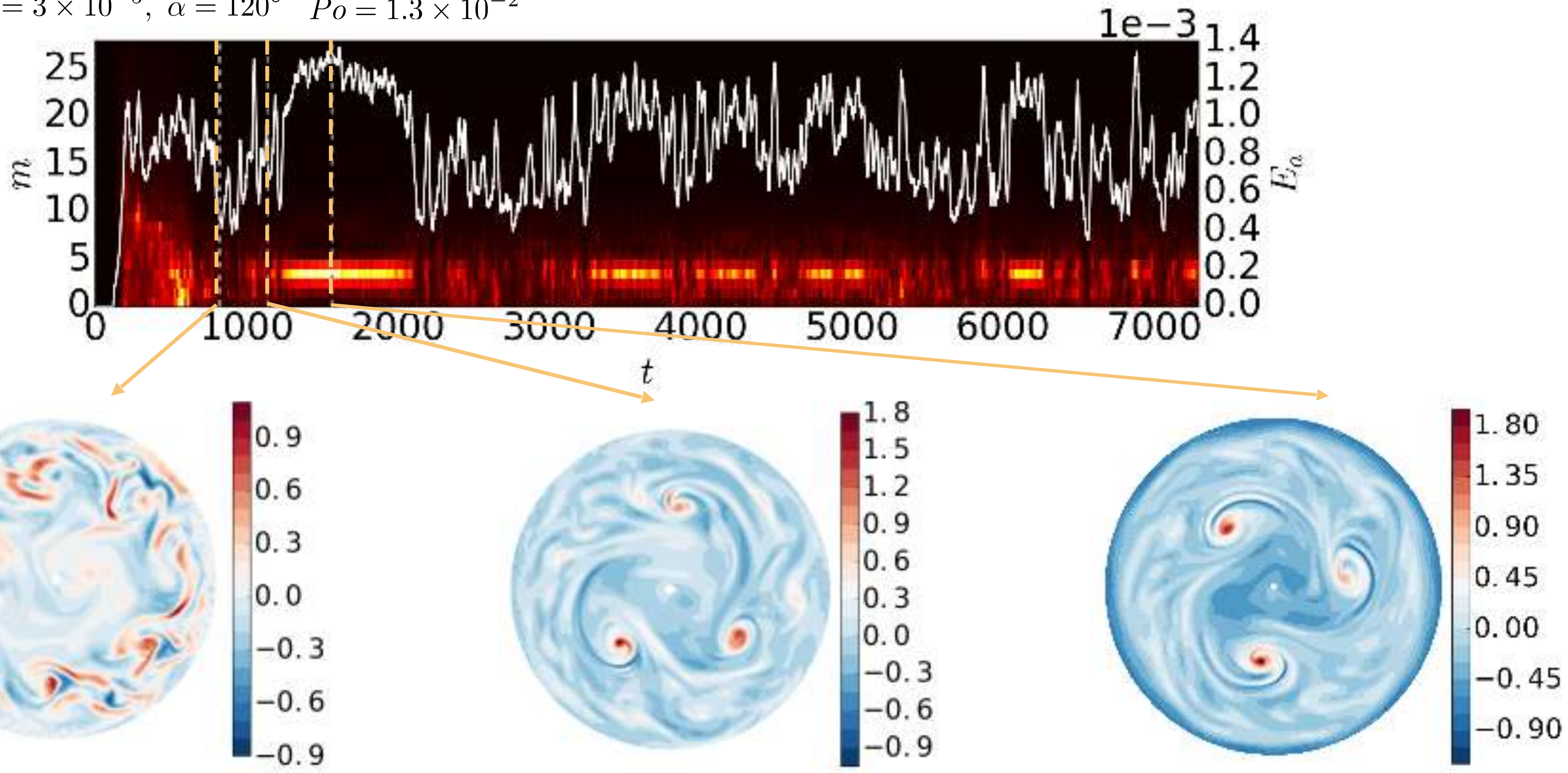
Time evolution of the anti-symmetric kinetic energy of each azimuthal mode m in the fluid frame (color map) and of the total anti-symmetric kinetic energy (White solid line). Fixed $\eta = 0.01$, $\alpha = 120°$ and $E = 3.0 \times 10^{-5}$. The Poincaré number increases from (a) $Po = 7 \times 10^{-3}$, (b) $Po = 8.5 \times 10^{-3}$ and (c) $Po = 1.3 \times 10^{-2}$, with dashed gray lines corresponding to the times of figure 10.

D. Cébron, R. Laguerre, J. Noir and N. Schaeffer.

“Precessing spherical shells : flows , dissipation , dynamo and the lunar core”, GJI, 2019

As previously reported by Lin et al., we observe large scale vortices in the system (LSV):

$\eta = 0.01,\ E = 3 \times 10^{-5},\ \alpha = 120^\circ \quad Po = 1.3 \times 10^{-2}$



PRL 119, [illegible] (2017) PHYSICAL REVIEW LETTERS

Inertial Wave Turbulence Driven by Elliptical Instability

Thomas Le Reun,[1] Benjamin Favier,[1] Adrian J. Barker,[2] and Michael Le Bars[1]
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For moderate Ekman numbers, and finite forcing amplitude typical of laboratory experiments and full 3D-DNS, the initial parametric instability evolves toward quasi geostrophic turbulence.

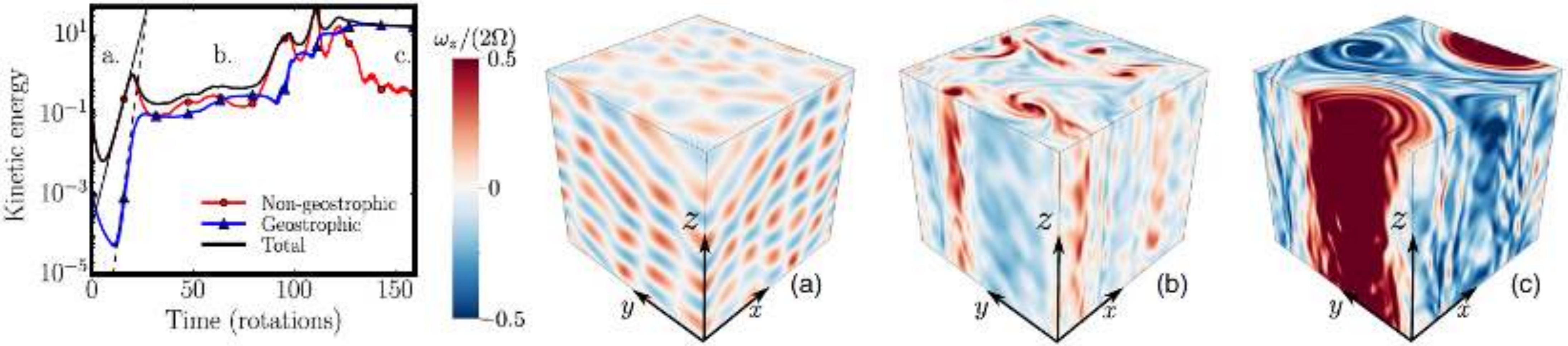
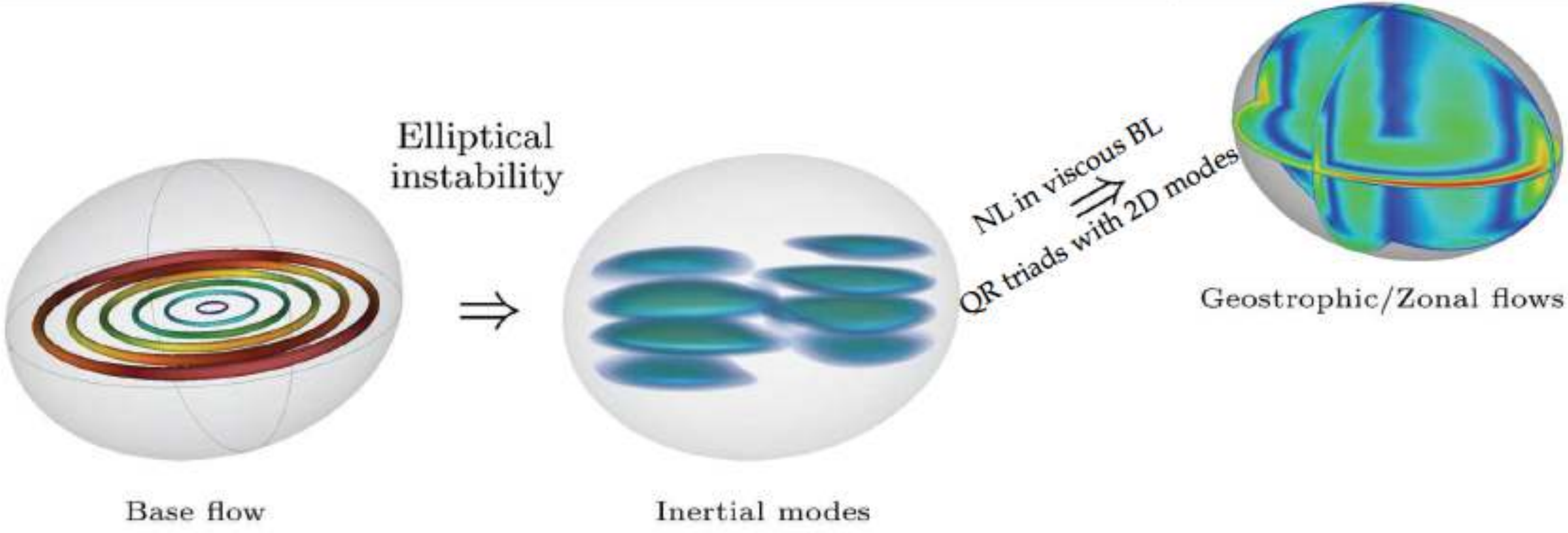


FIG. 1. Left: typical evolution of the volume-averaged kinetic energy for $E = 10^{-5}$ and a 256^3 resolution from the exponential growth of a few waves (a) to its nonlinear saturation (b) and (c). Continuous and dashed lines account for exponential growth with rates 2σ and 4σ, respectively, with σ the theoretical viscous growth rate of the instability. The velocity amplitudes are normalized by $k_{\mathrm{res}}^{-1}\Omega$. Right: corresponding typical snapshots of the vertical vorticity normalized by the background vorticity.

The nonlinear fate of the elliptical instability



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For small Ekman numbers, and small forcing amplitude typical of planetary conditions, it is suggested that the initial parametric instability evolves small scales wave turbulence through a series of consecutive triadic resonance.

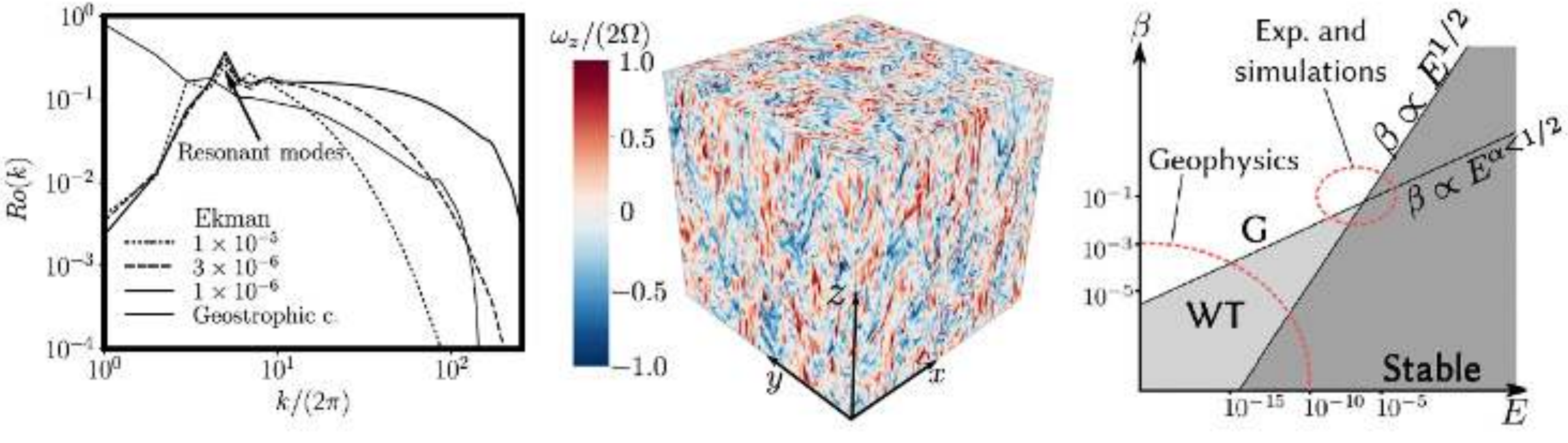

FIG. 4. Left: scale dependence of Rossby number for $\beta = 5\times10^{-2}$ and $f_r = 1$ for $E = 10^{-5}$, 3×10^{-6}, 10^{-6}; results from the geostrophic saturation [Fig. 1(c)] are shown for comparison. Center: vertical vorticity in the $E = 10^{-6}$ case. Right: schematic diagram illustrating where the geostrophic and wave turbulence regimes can be expected, depending on β and E. WT and G stand for wave turbulence and geostrophic types of saturation, respectively.

The nonlinear fate of the elliptical instability

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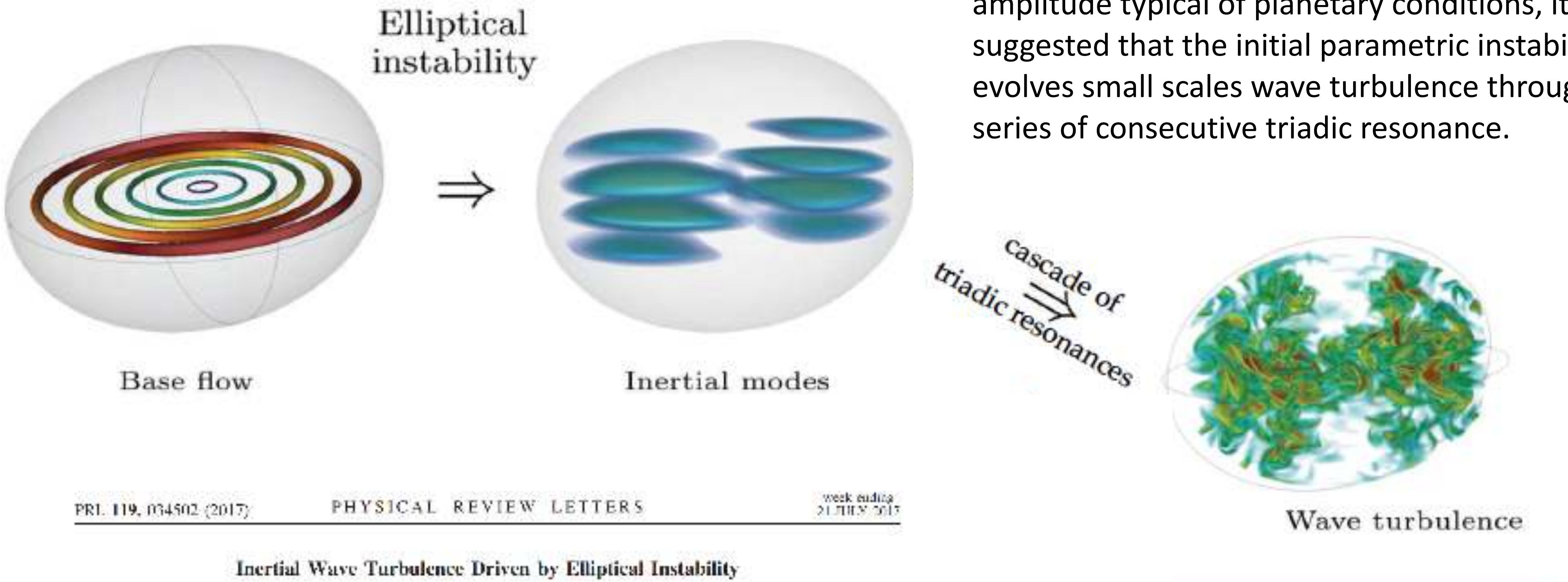

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To go further

Nonlinear evolution of the elliptical instability: an example of inertial wave breakdown
D M Mason; RR Kerswell - 1999 - Journal of Fluid Mechanics
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Elliptical instability
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Elliptical instabilities of stratified, hydromagnetic waves
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Abstract: ...December 1992) ... 106 R . R . **KERSWELL elliptical** flow which should be set up in ...
Authors: R R **Kerswell**

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Abstract: ...is therefore expected in the **elliptically** distorted) when
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The Instability Of Precessing Flow
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Thomas Le Reun; B Favier; AJ Barker; ... - 2017 - Physical Review Letters
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Inertial wave turbulence driven by elliptical instability
Thomas Le Reun; B Favier; AJ Barker; ...
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Non-linear evolution of tidally forced inertial waves in rotating fluid bodies
B. Favier; AJ Barker; C Baruteau; GI ... - 2014 - Monthly Notices of the Royal Astronomical Society
Authors: B. Favier, A. J. **Barker**, C. Baruteau, G. I. Ogilvie

Nonlinear tides in a homogeneous rotating planet or star : global simulations of the elliptical instability
Adrian J Barker - 2016
Authors: Adrian J **Barker**