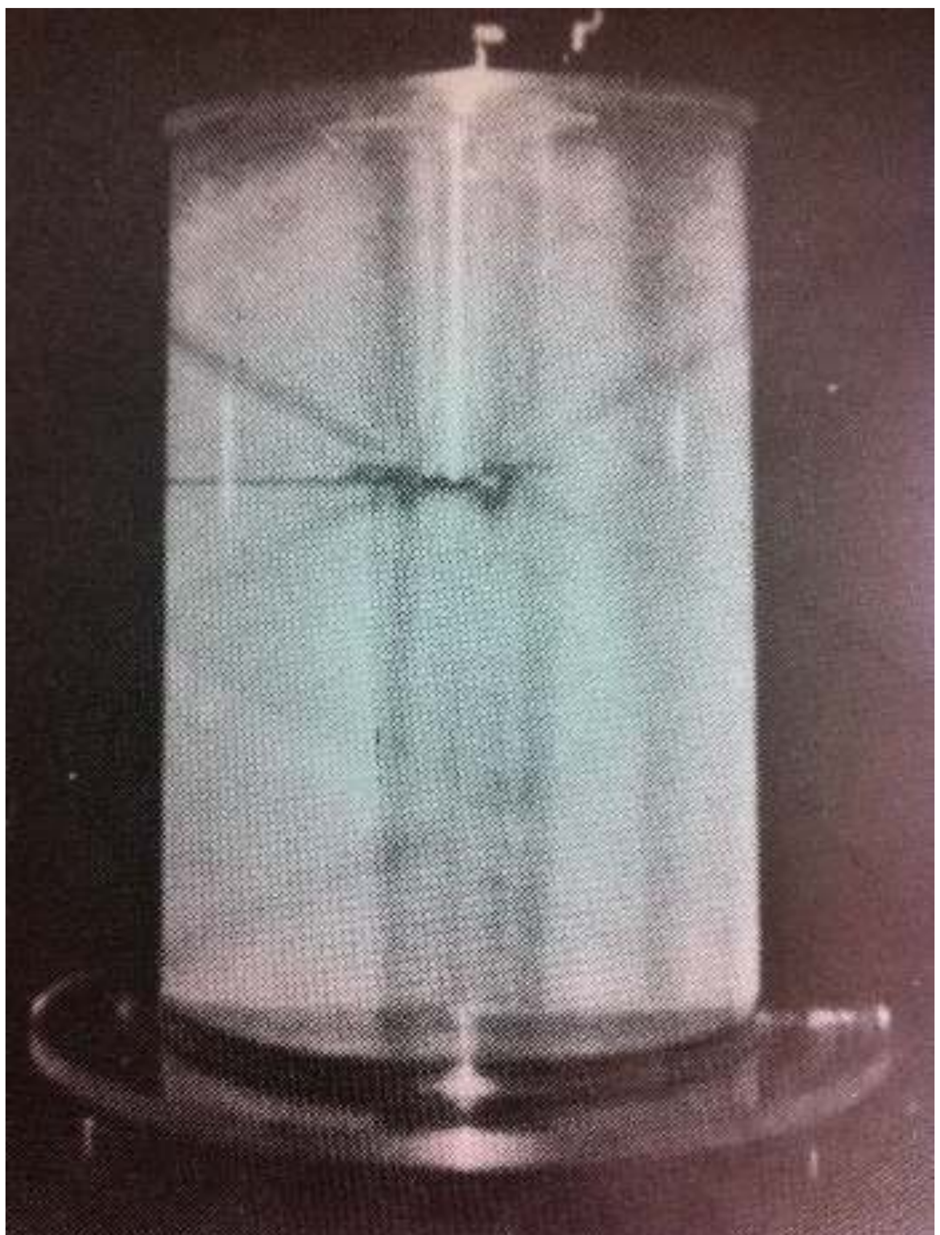
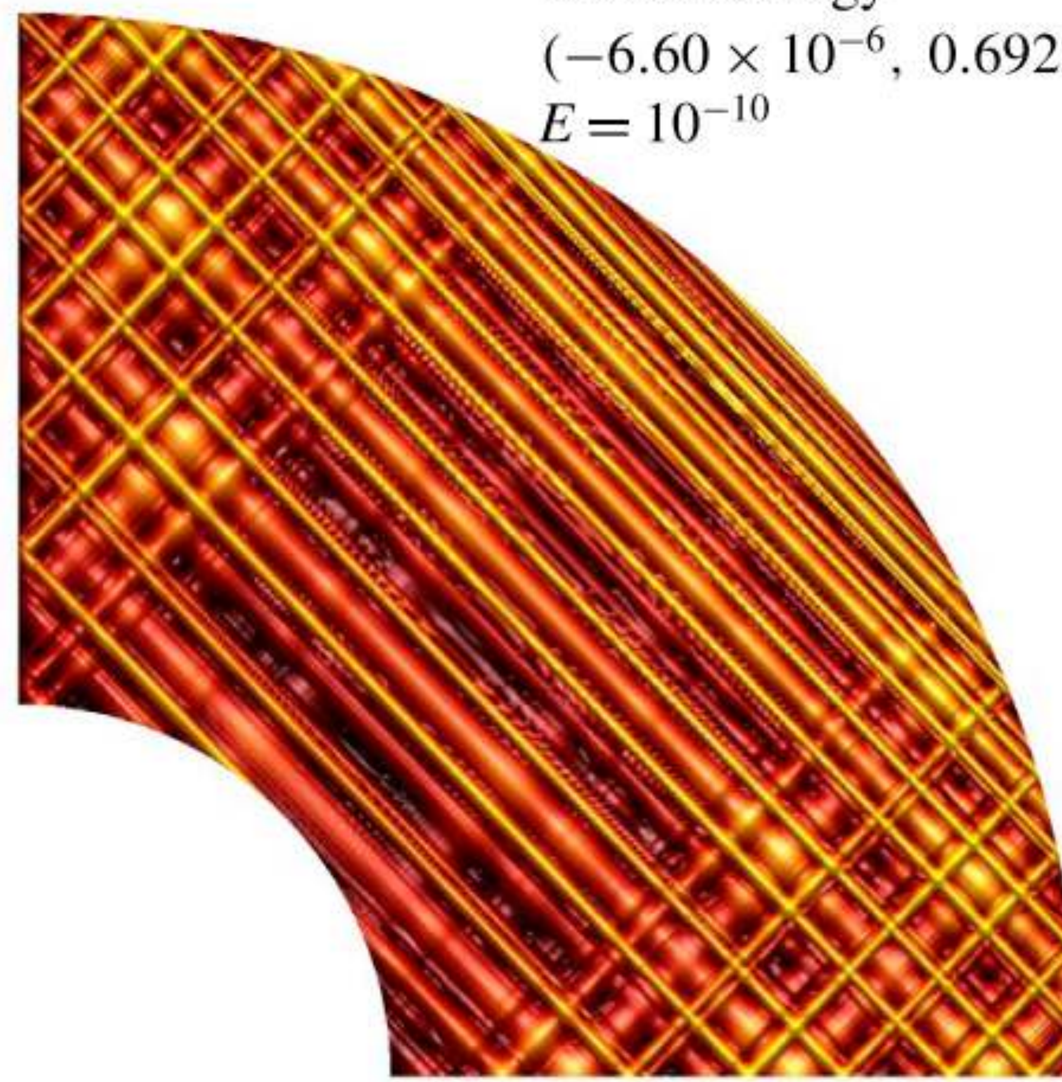
# Part II:Inertial Waves and Inertial Modes

Lecture by J. Noir at WITGAF Cargese, Corsica, 2019



Experiment by Goertler 1957, picture from: The theory of rotating fluids by H. P. Greenspan 1968

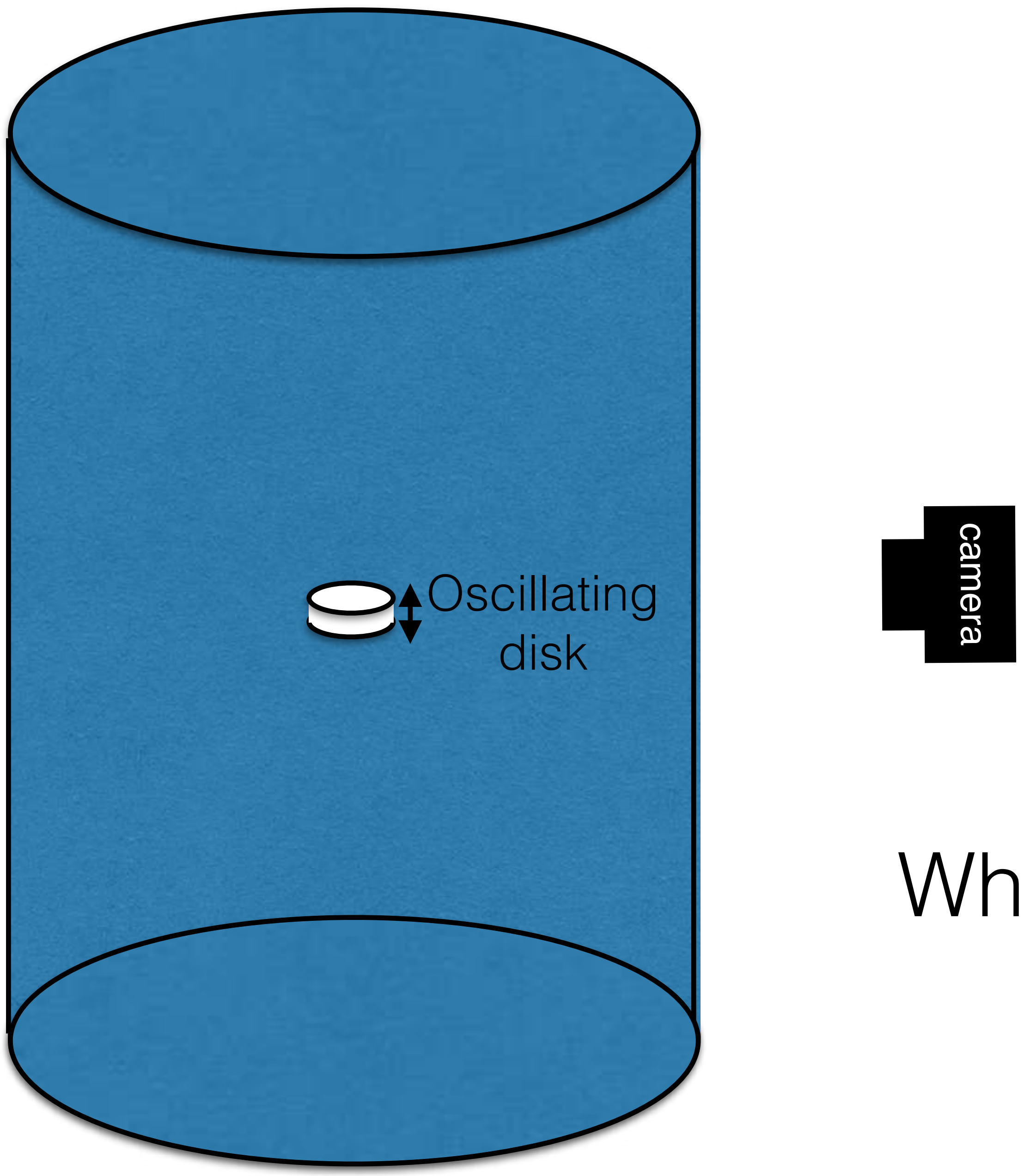



Axisymmetric inertial modes in a spherical shell at low Ekman numbers: M. Rieutord and L Valdettaro, 2018

1. Some observations.
2. Inviscid Inertial Waves.
3. Inviscid Inertial Modes.
4. Viscous Correction.
5. Resonances, large flows from small perturbations.

# What makes rotating fluids so special…

Let's now consider another example.
We still consider the case of a small perturbation but this time it is an oscillatory forcing




- Let's first consider a cylindrical tank filled with water that is **not rotating.**
- At the centre we put a small disk that oscillates vertically at a frequency $\omega_1$.
- The camera rotates with the cylinder

What do you expect to see from the camera ?

# What makes rotating fluids so special…

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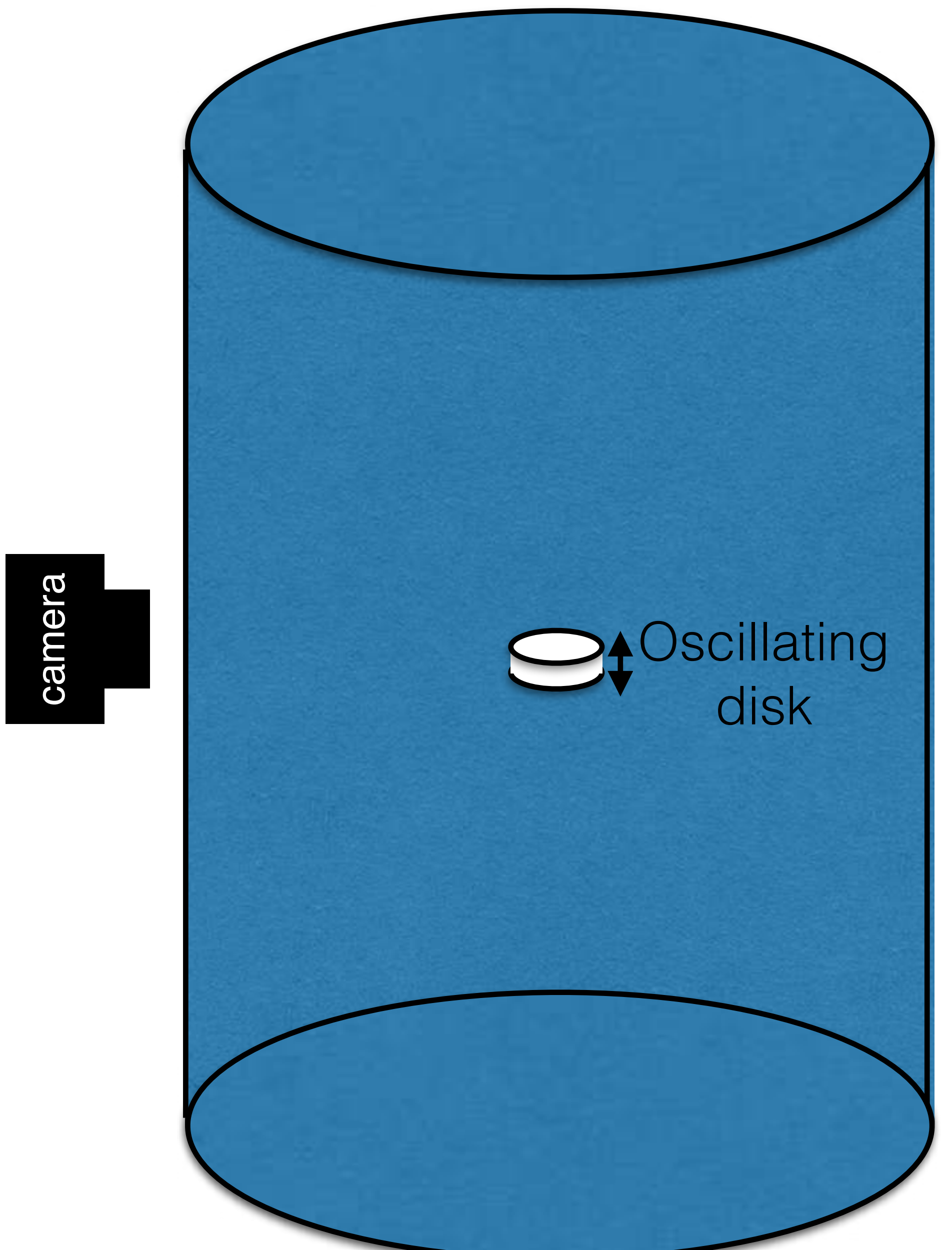
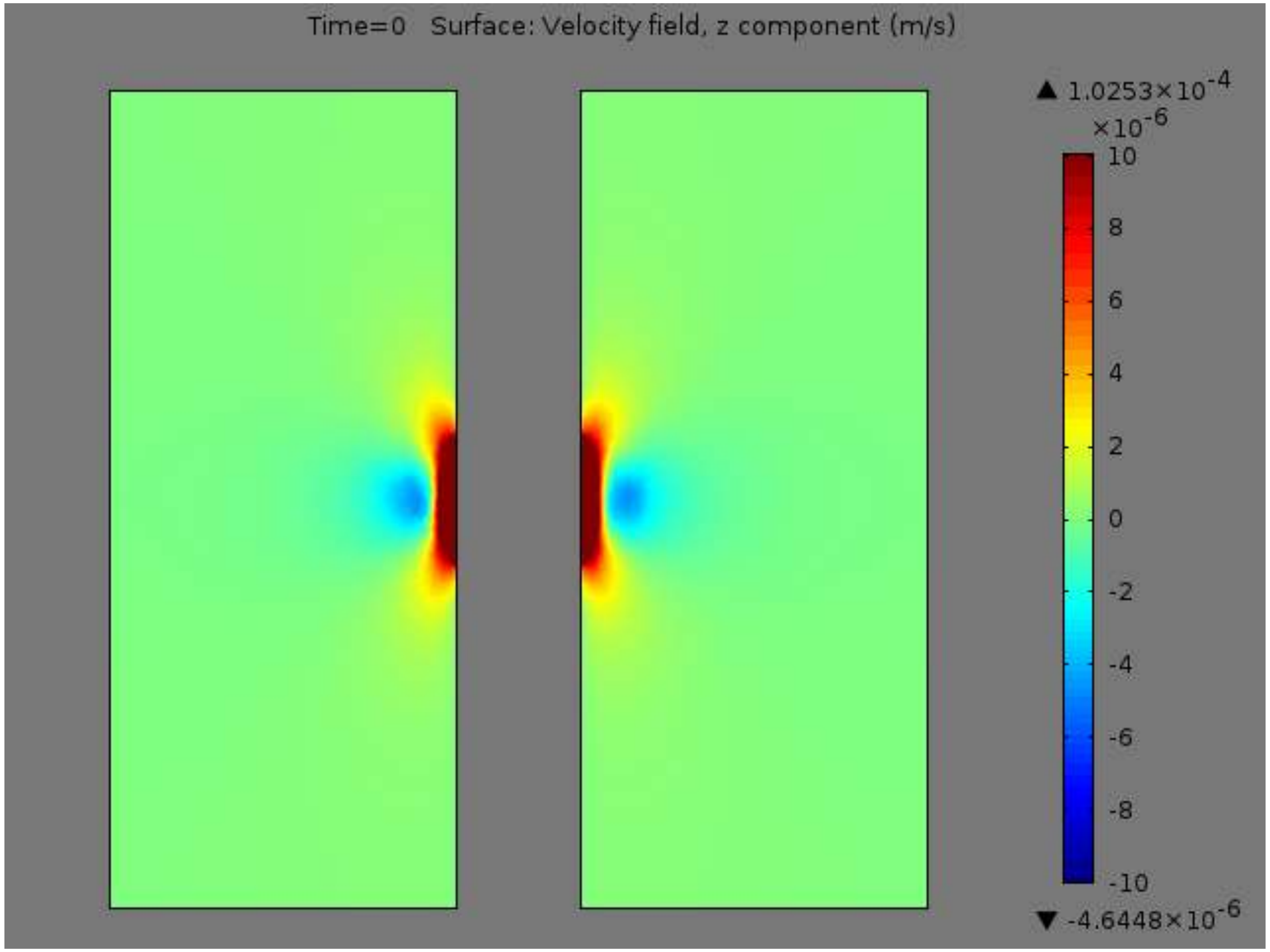
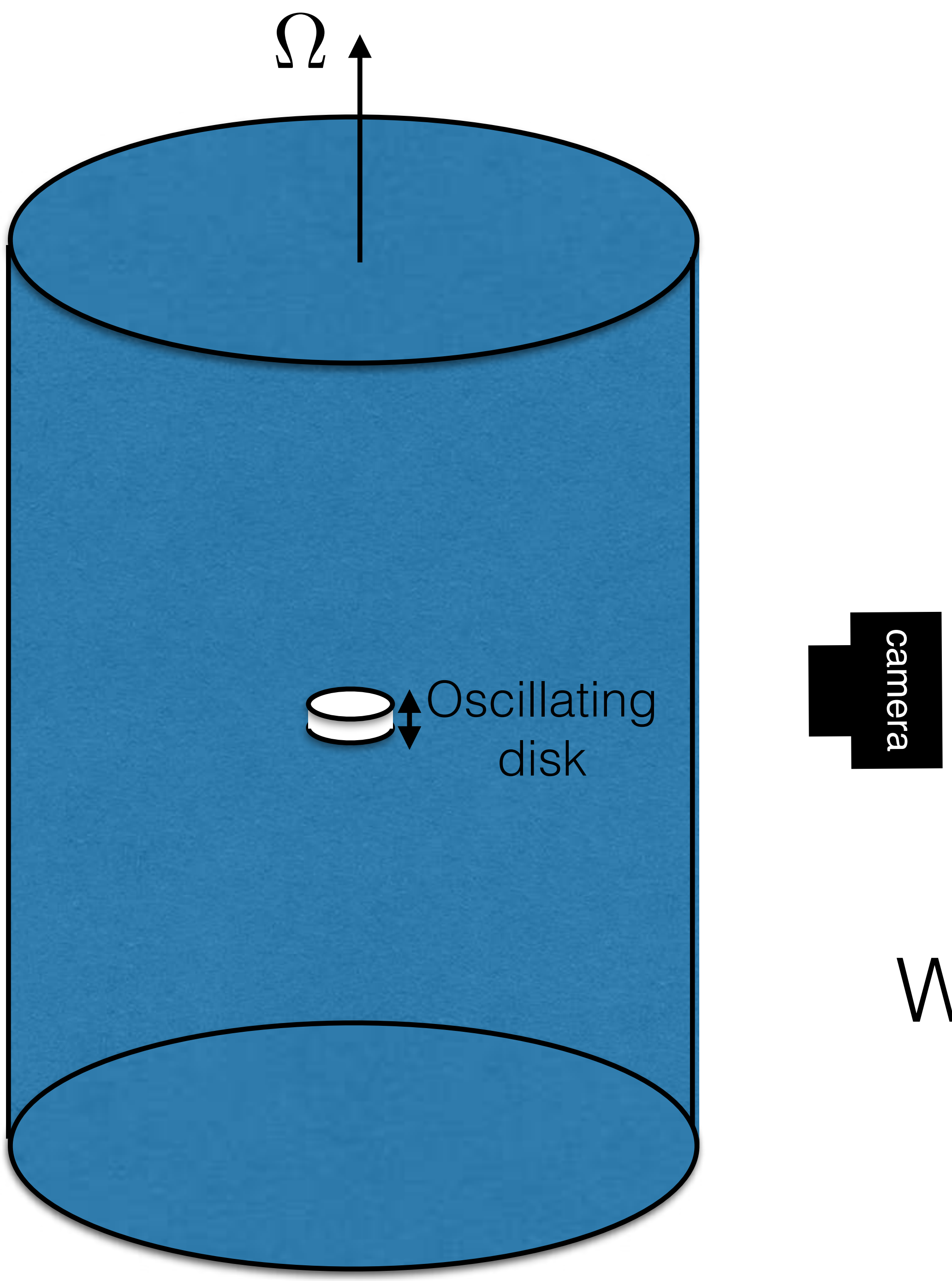


# What makes rotating fluids so special…

Let's now consider another example.
We still consider the case of a small perturbation but this time it is an oscillatory forcing




- Let's now consider a cylindrical tank filled with water that is **rotating at Ωo**.
- At the centre we put a small disk that oscillates vertically at a frequency $\omega_1$.
- The camera rotates with the cylinder

What do you expect to see from the camera ?

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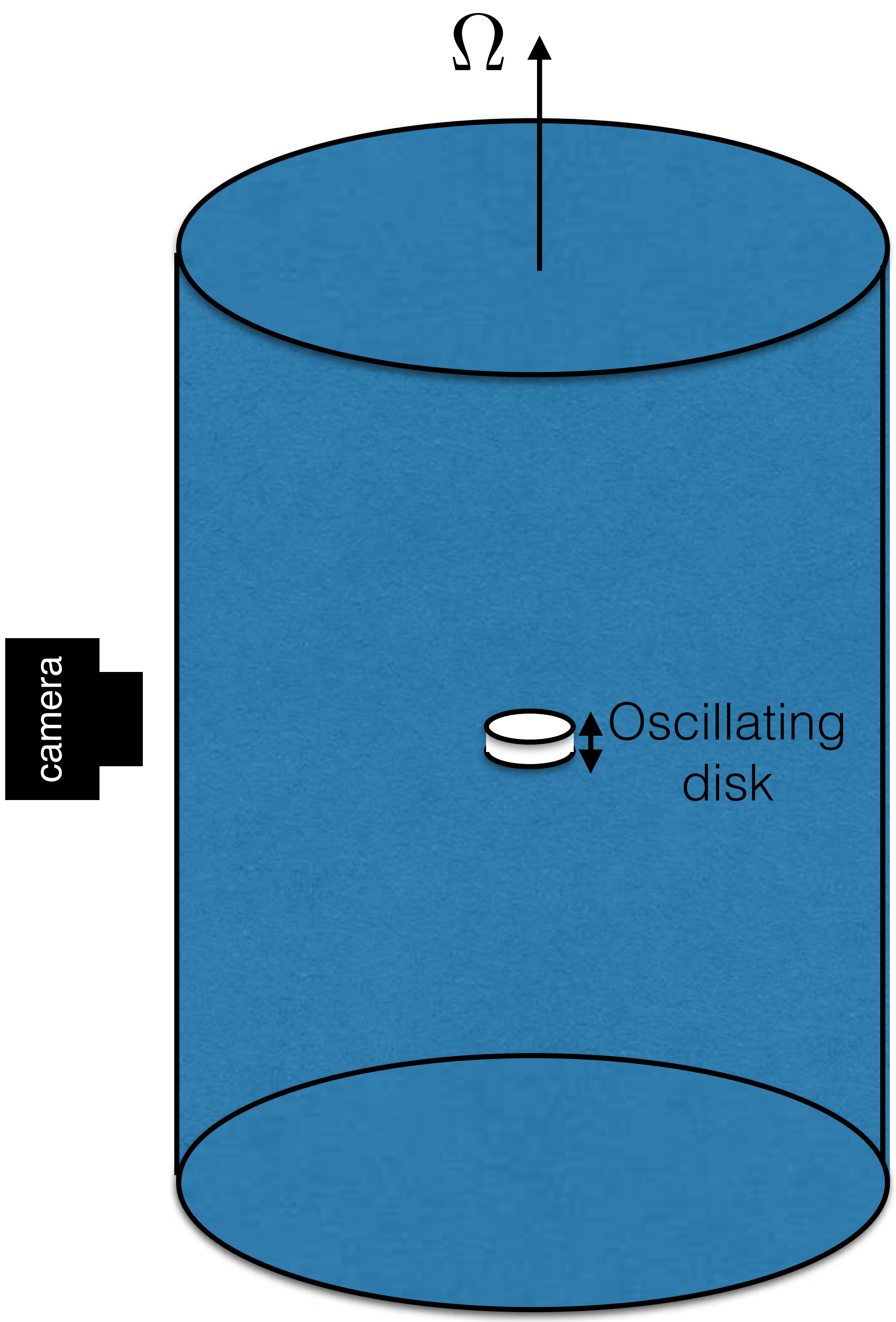
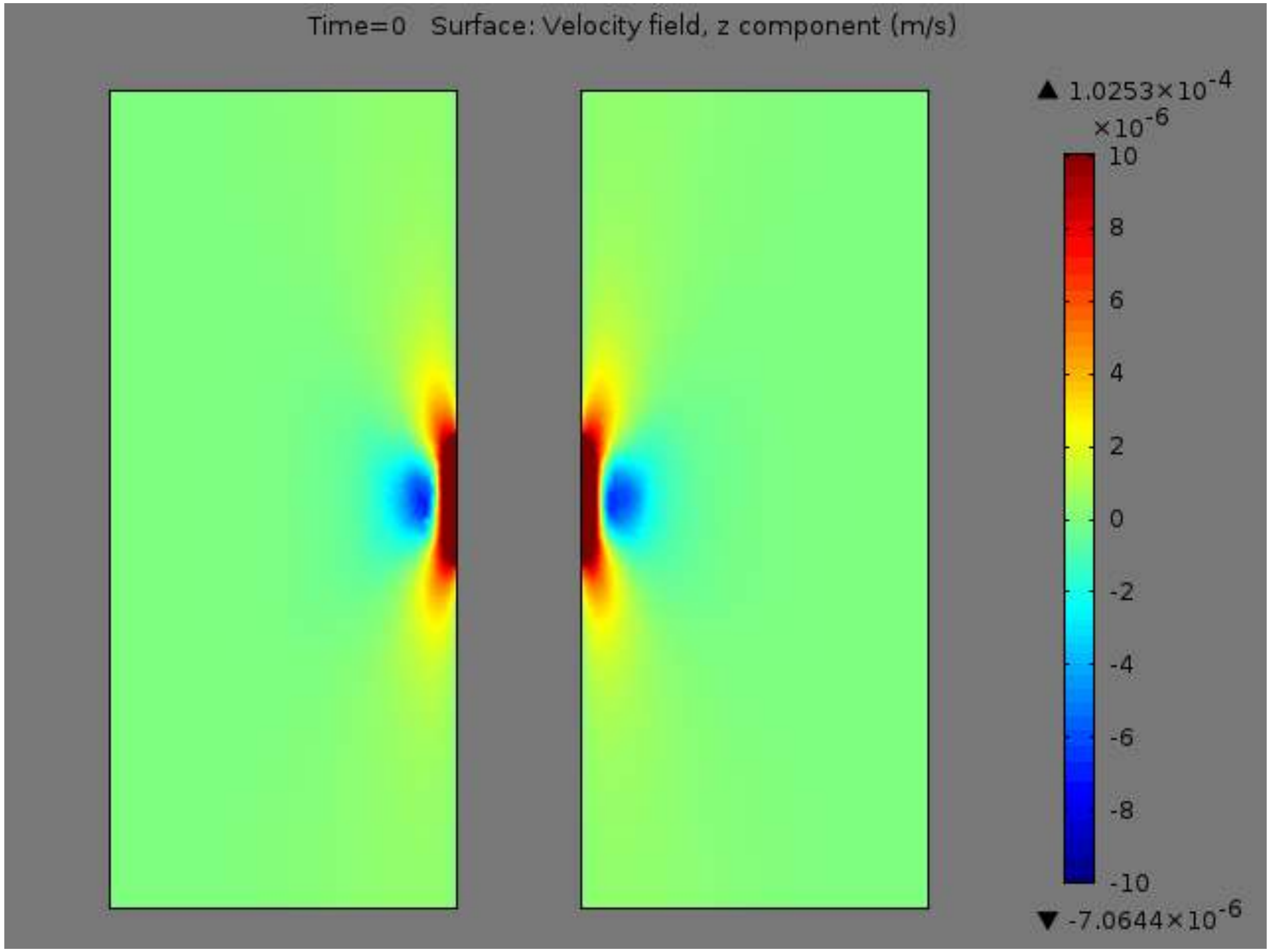
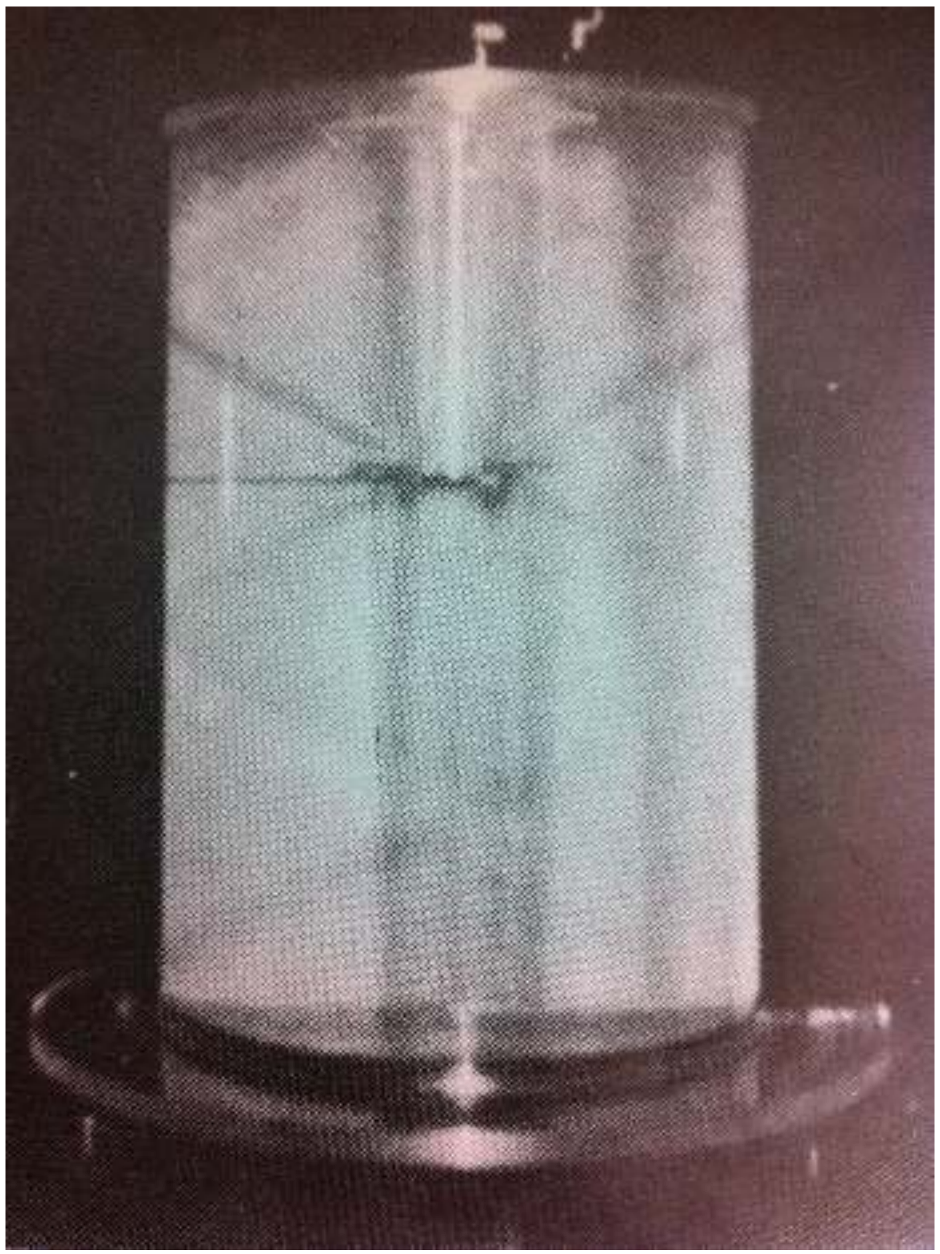


# What makes rotating fluids so special…

Let's now consider another example of perturbations of a rapidly rotating fluid.
We still consider the case of a small perturbation but this time it is an oscillatory forcing



Experiment by Goertler 1957, picture from: The theory of rotating fluids by H. P. Greenspan 1968

- A rapidly rotating fluid system can support waves propagation, the so-called inertial waves.
- In a bounded fluid, like a cylinder the waves can from constructive interferences and forms Inertial modes.

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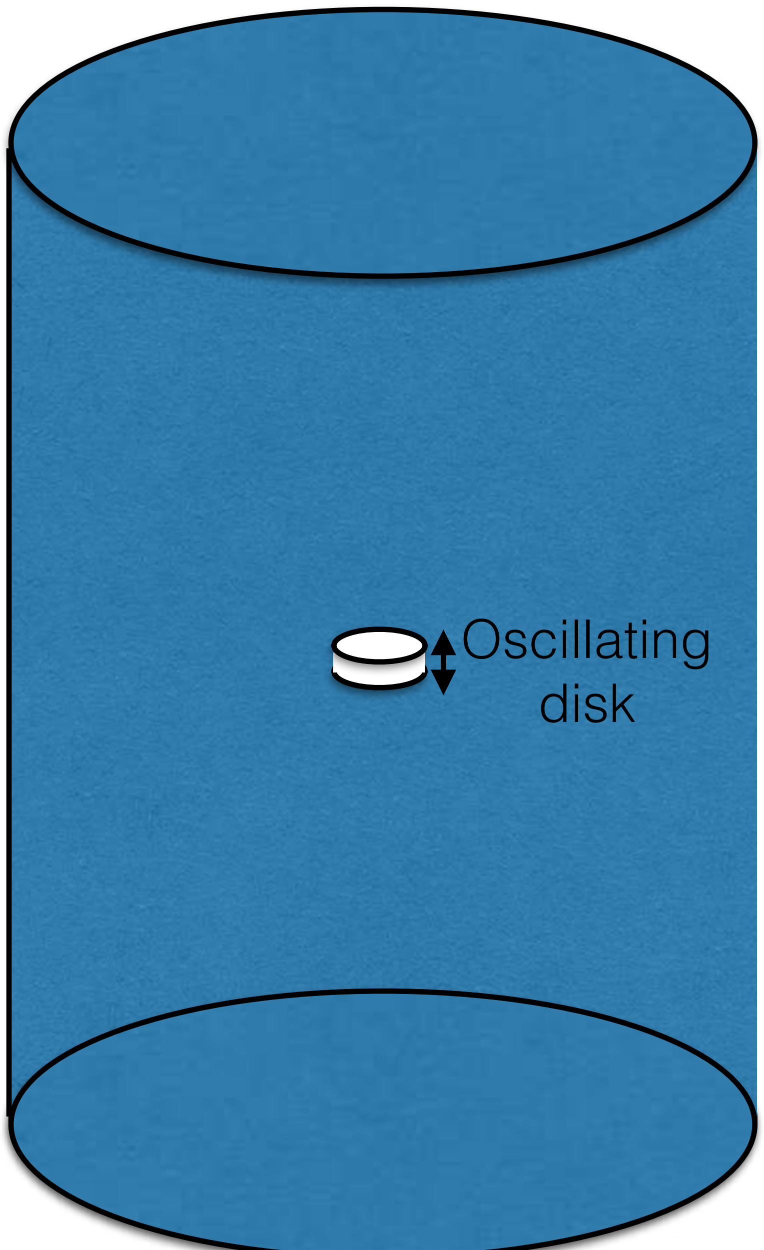
# The non rotating inviscid case in a closed container

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}\Pi + \frac{1}{Re}\vec{\nabla}^2\vec{u} + \vec{f}$$

$$\vec{\nabla}\cdot\vec{u} = 0, \quad \hat{\mathbf{n}}\cdot\vec{u} = 0|_S$$




**<u>Let's consider the case</u>**

No external force

no rotation

low viscosity fluid

small oscillations

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\Pi$$

# The non rotating case in a closed container

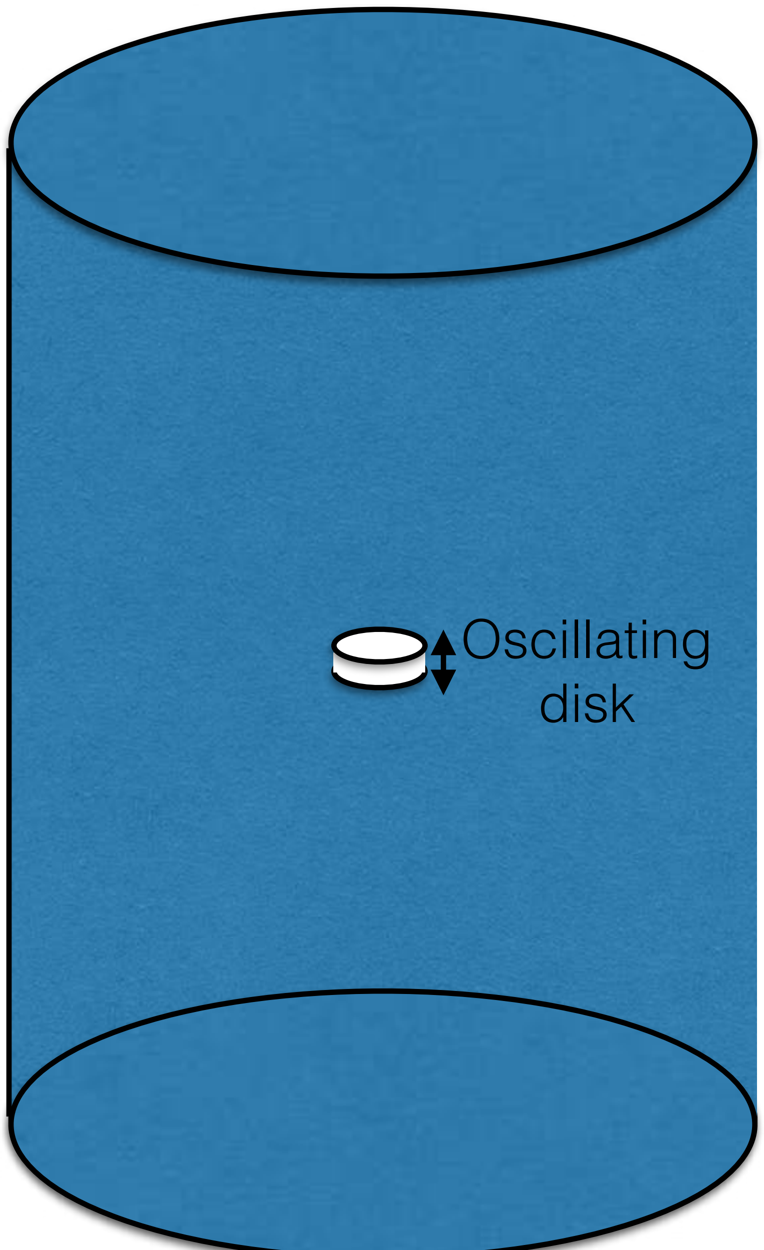
$$\vec{\nabla} \cdot \left[\frac{\partial \vec{u}}{\partial t}\right] = -\vec{\nabla} \cdot \left[\vec{\nabla}\Pi\right]$$

$$\vec{\nabla} \cdot \vec{u} = 0, \quad \hat{\mathbf{n}} \cdot \vec{u} = 0|_S$$

$$\vec{\nabla}^2\Pi = 0 \quad \hat{\mathbf{n}} \cdot \vec{\nabla}\Pi = 0$$

In the volume    On the surface

$$\int_S \Phi\hat{\mathbf{n}} \cdot \vec{\nabla}\Psi dS = \int_V \left(\Phi\vec{\nabla}^2\Psi + \vec{\nabla}\Phi \cdot \vec{\nabla}\Psi\right) dV$$



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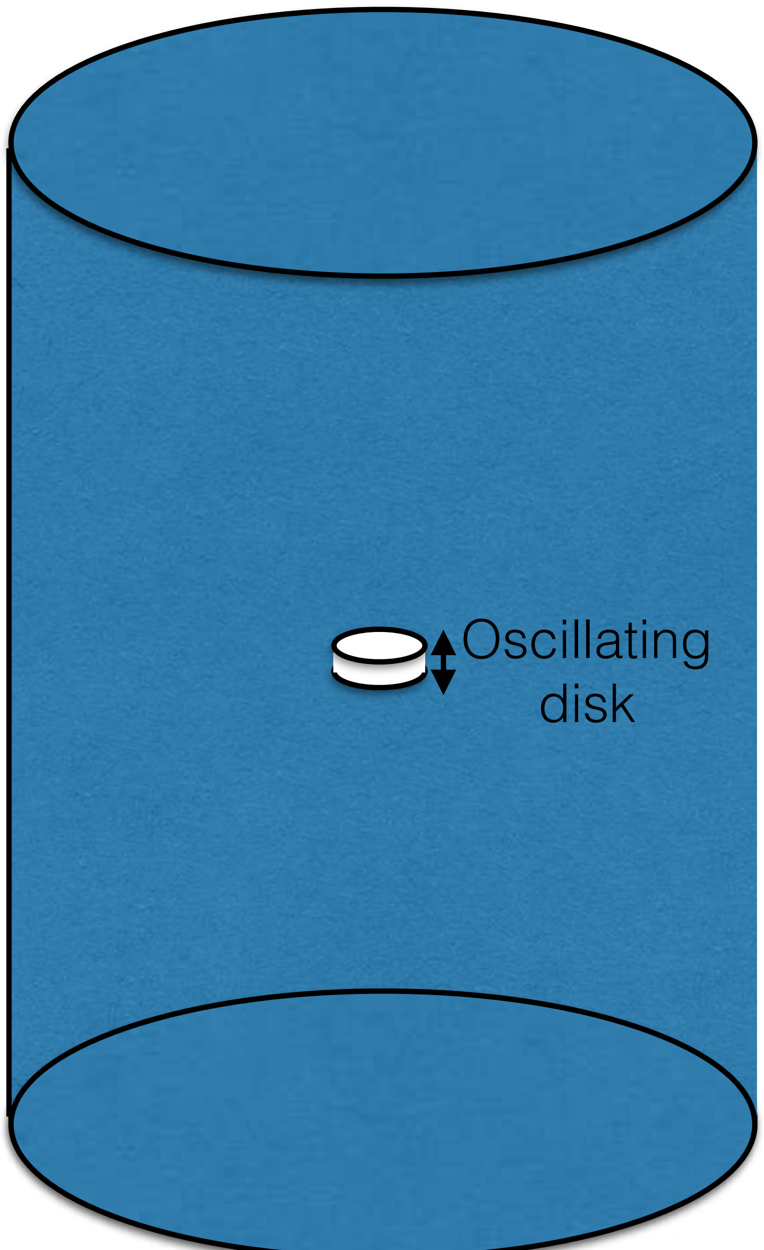



$$\int_S \Phi \hat{\mathbf{n}} \cdot \vec{\nabla}\Psi dS = \int_V \left(\Phi \vec{\nabla}^2 \Psi + \vec{\nabla}\Phi \cdot \vec{\nabla}\Psi\right) dV$$

$$\int_S \Pi \hat{\mathbf{n}} \cdot \vec{\nabla}\Pi dS = \int_V \left(\Pi \vec{\nabla}^2 \Pi + \vec{\nabla}\Pi \cdot \vec{\nabla}\Pi\right) dV$$

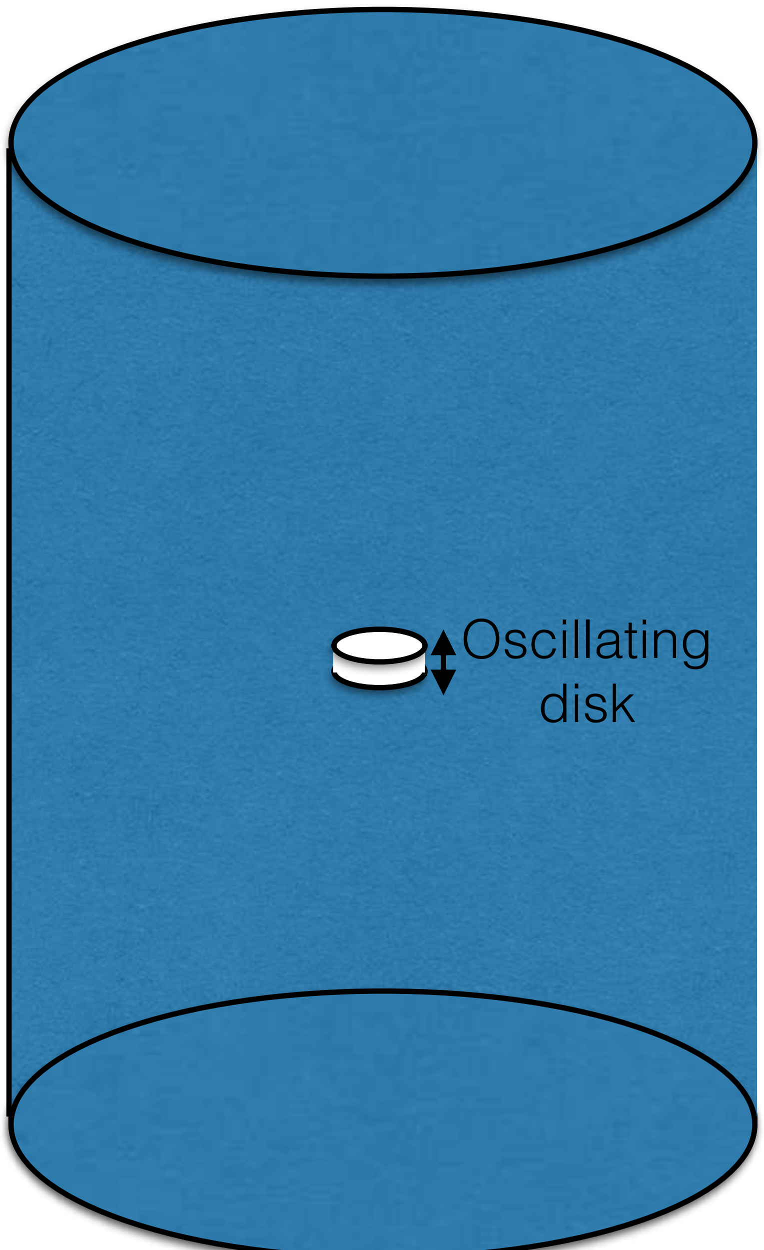
# The non rotating case in a closed container

$$\vec{\nabla}\cdot\left[\frac{\partial \vec{u}}{\partial t}\right] = -\vec{\nabla}\cdot\left[\vec{\nabla}\Pi\right]$$

$$\vec{\nabla}\cdot\vec{u}=0, \quad \hat{\mathbf{n}}\cdot\vec{u}=0|_S$$

$$\vec{\nabla}^2\Pi = 0 \qquad \hat{\mathbf{n}}\cdot\vec{\nabla}\Pi = 0$$

In the volume — On the surface




$$\int_S \Phi\hat{\mathbf{n}}\cdot\vec{\nabla}\Psi dS = \int_V \left(\Phi\vec{\nabla}^2\Psi + \vec{\nabla}\Phi\cdot\vec{\nabla}\Psi\right) dV$$

$$\int_S \Pi\hat{\mathbf{n}}\cdot\vec{\nabla}\Pi dS = \int_V \left(\Pi\vec{\nabla}^2\Pi + \vec{\nabla}\Pi\cdot\vec{\nabla}\Pi\right) dV$$

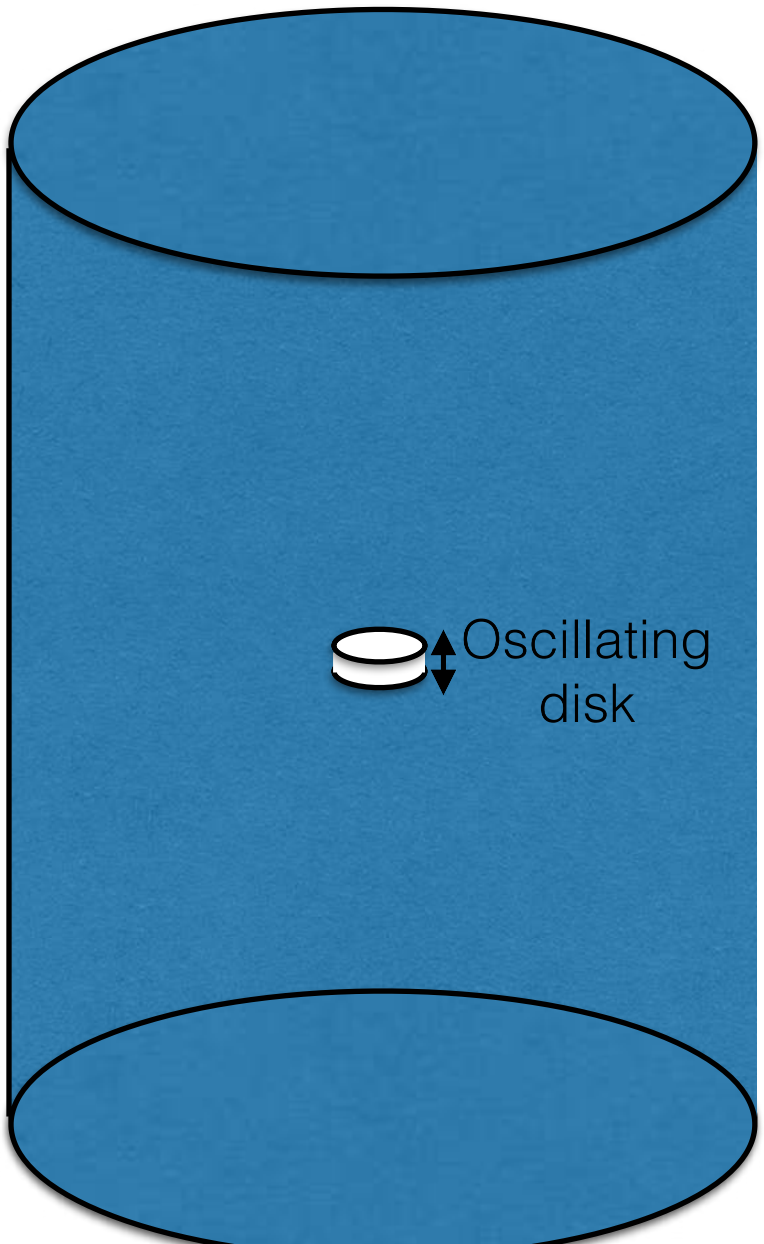
$$0 = \int_V \vec{\nabla}\Pi^2 dV \longrightarrow \vec{\nabla}\Pi \equiv 0$$

In the volume

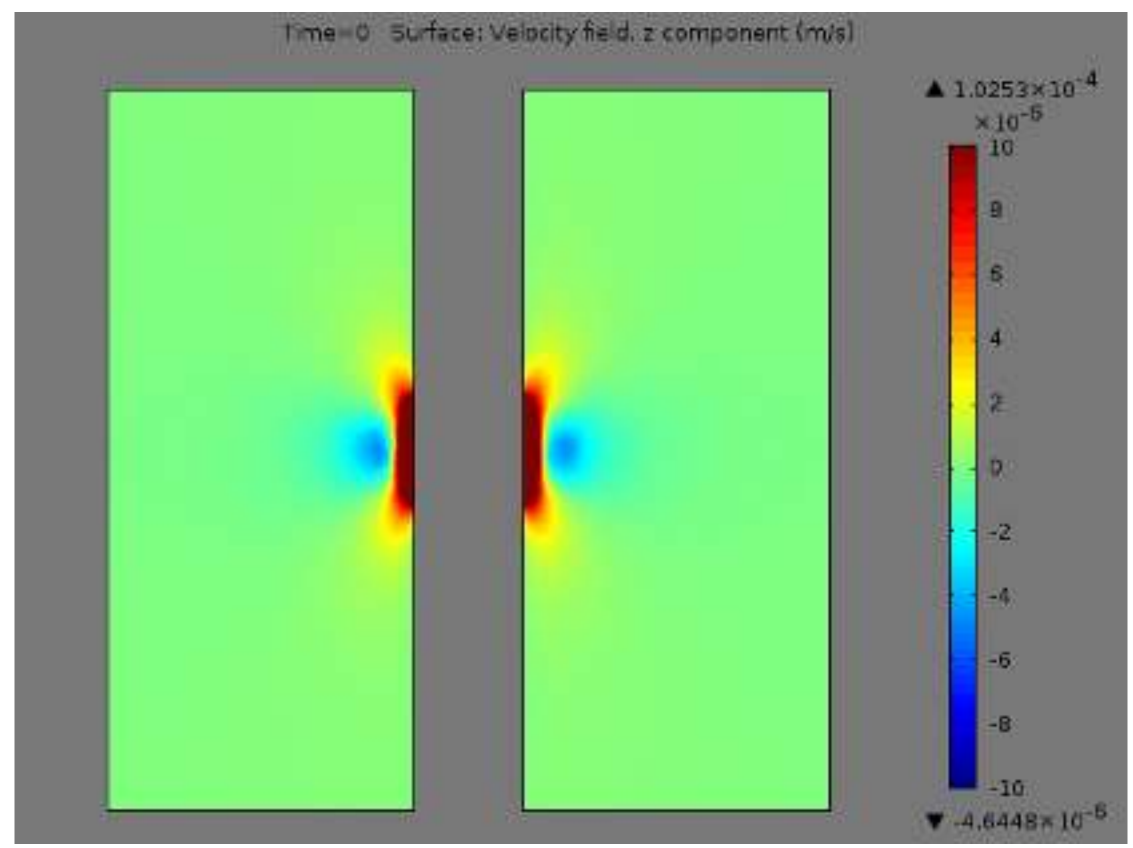
# The non rotating case in a closed container

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\Pi \equiv 0$$

$$\vec{\nabla} \cdot \vec{u} = 0, \ \ \hat{\mathbf{n}} \cdot \vec{u} = 0|_S$$

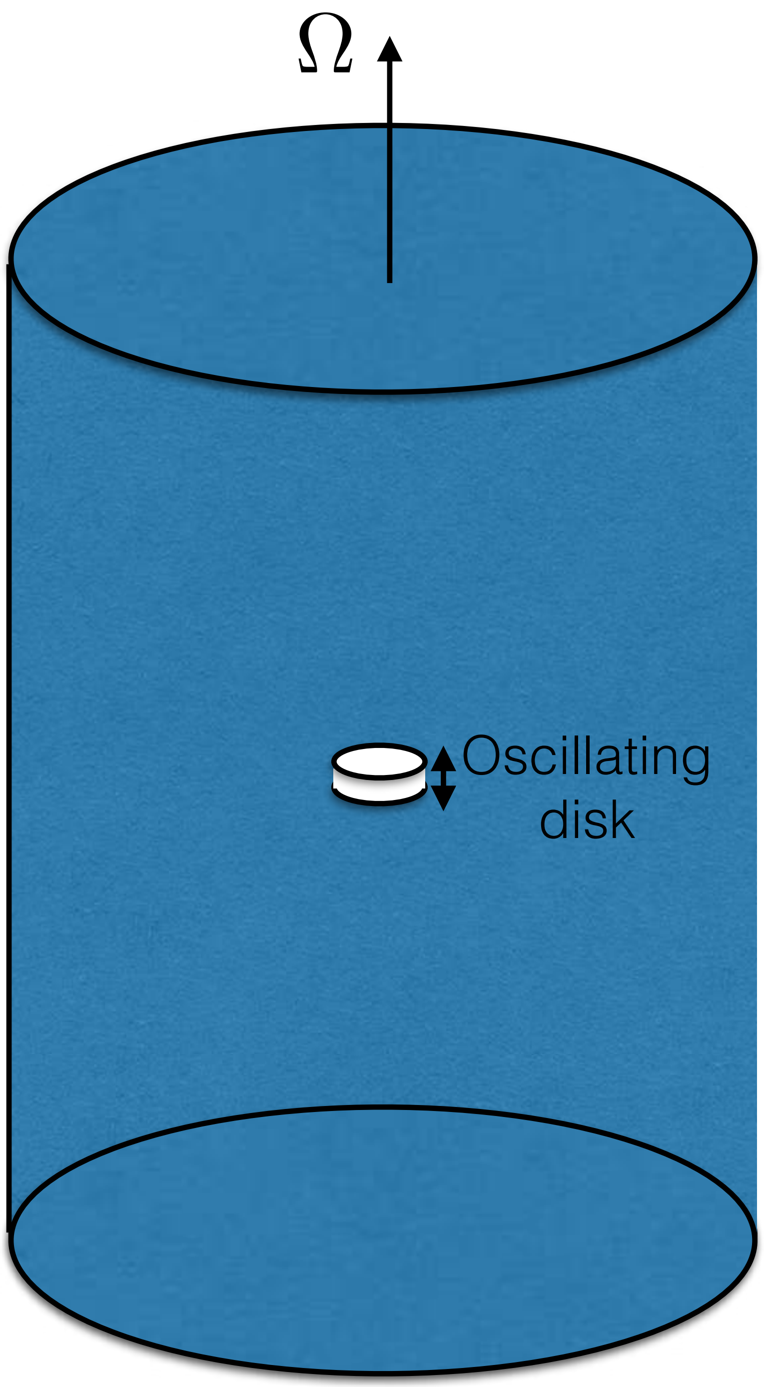



**Non-rotating, inviscid and neutrally buoyant fluids cannot sustain inertial oscillations. What we observe in the numerical simulations is the viscous diffusion of momentum, the stokes layer.**



# Inviscid Inertial waves equation:

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} \Pi + E \vec{\nabla}^2 \vec{u} + \vec{r} \times \frac{d\vec{\Omega}}{dt} + \vec{f}$$




**<u>Let's consider the case</u>**

No external force

constant rapid rotation

low viscosity fluid

small oscillations ($u \ll \Omega r$)

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla} \Pi$$

# The rotating case: Inertial waves

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\vec{u}(t, \vec{r}) = \vec{u}(\vec{r}) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\Pi(t, \vec{r}) = \pi(\vec{r}) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Substitution of this particular form of solutions in the unforced equations leads to:

- From the continuity equation we deduce that the velocity is perpendicular to the wave vector:

$$\vec{u} \cdot \vec{k} = 0$$

- From the Navier-Stokes equation we deduce that the angle between the wave vector and the rotation vector is given by the dispersion relation:

$$\omega = \pm 2\vec{\Omega} \cdot \frac{\vec{k}}{|\vec{k}|} = \pm 2\Omega \cos\theta$$

Where, $\theta$ is the angle between the wave vector and the rotation axis. Wave like solution exist only for frequencies

$$0 < |\frac{\omega}{\Omega}| < 2$$

# Inertial waves properties

Like any other waves, inertial waves have a phase and group velocity

## Phase velocity

Velocity at which one should propagate to see a constant phase, $\mathbf{k}\cdot\mathbf{r} - \omega t = cte$

$$\mathbf{V}_\phi = 2\frac{\mathbf{\Omega}\cdot\mathbf{k}}{|\mathbf{k}|^3} = 2\Omega\cos(\theta)\frac{\mathbf{k}}{|\mathbf{k}|^3} \tag{100}$$

The phase velocity is along $\mathbf{k}$. No energy, i.e. information, propagates with $\mathbf{V}_\phi$

## Group velocity

Velocity at which energy, i.e. information, propagates in the system.

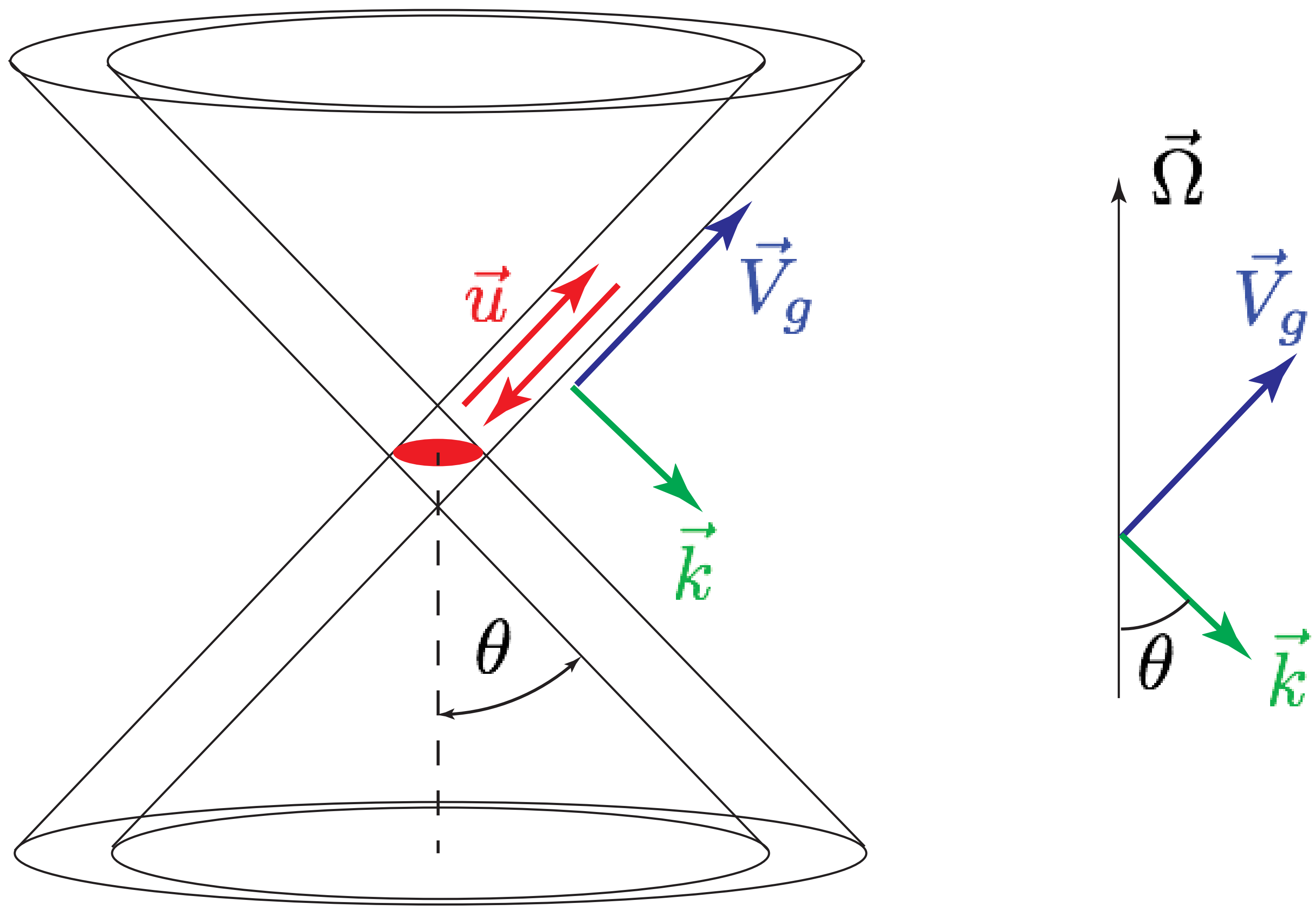
$$\mathbf{V}_g = 2\frac{\mathbf{k}\times(\mathbf{\Omega}\times\mathbf{k})}{|\mathbf{k}|^3} \;,\; |V_g| = 2\frac{\Omega}{|\mathbf{k}|}\sin\theta$$

- The energy propagates perpendicularly to the phase velocity, i.e. to the wave vector.
- The energy propagates faster for small wave numbers, i.e. large scale perturbations.
- Over one oscillation of the wave, the energy propagates over a typical length scale $L = \mathbf{V}/\omega = \lambda\frac{\Omega}{\omega}$ with $\lambda = 1/k$ the wave length.

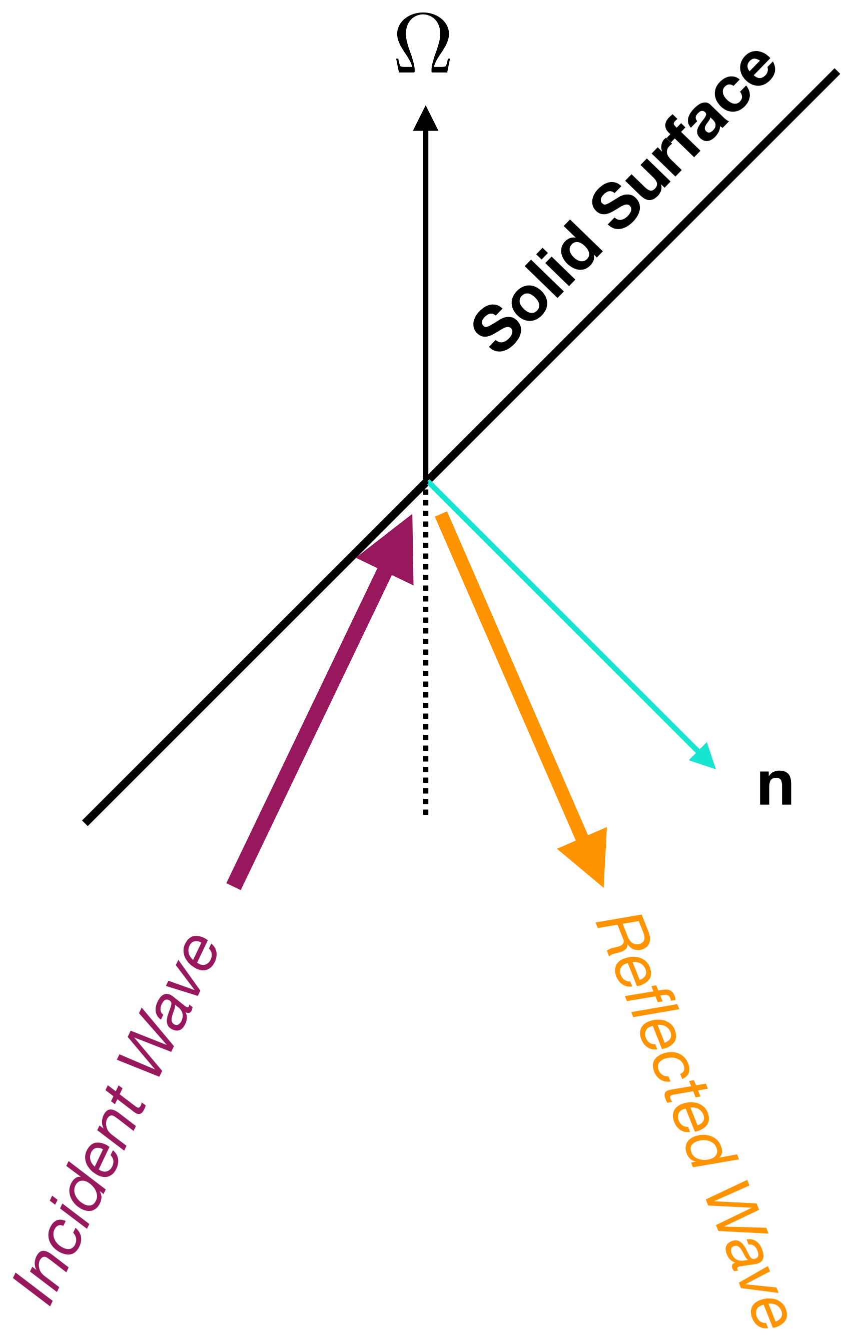
# Inertial waves propagation from a localized perturbation

Let's imagine a velocity perturbation, here in red, the gradient of velocity propagates along cones with a semi-aperture angle $\theta$ given by the dispersion relation:

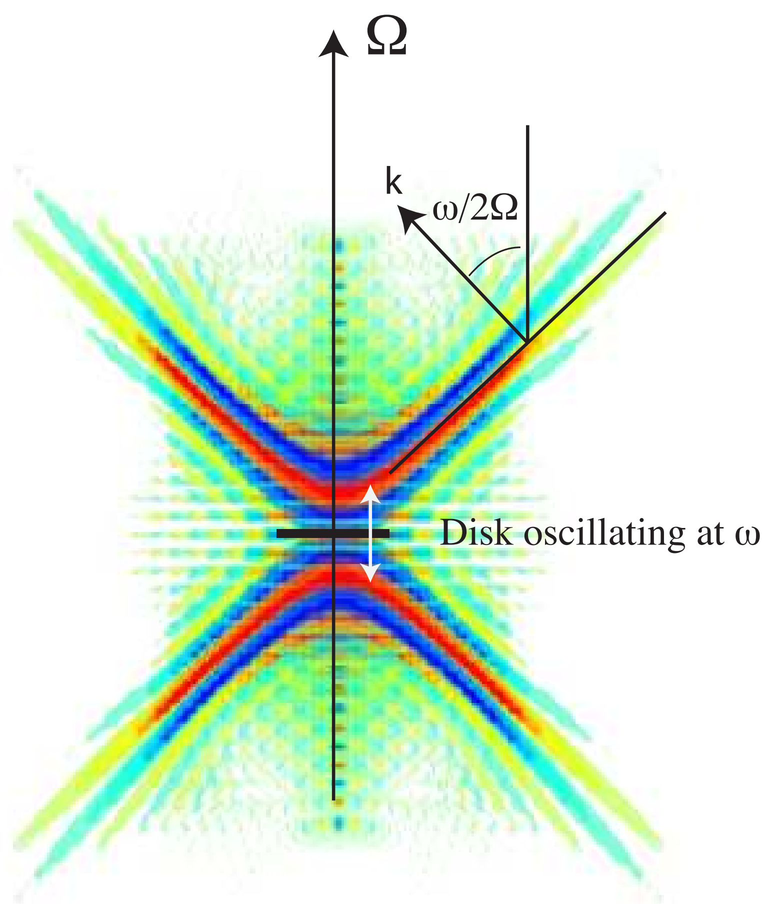
$$\frac{\omega}{\Omega} = \pm 2 \cos(\theta)$$



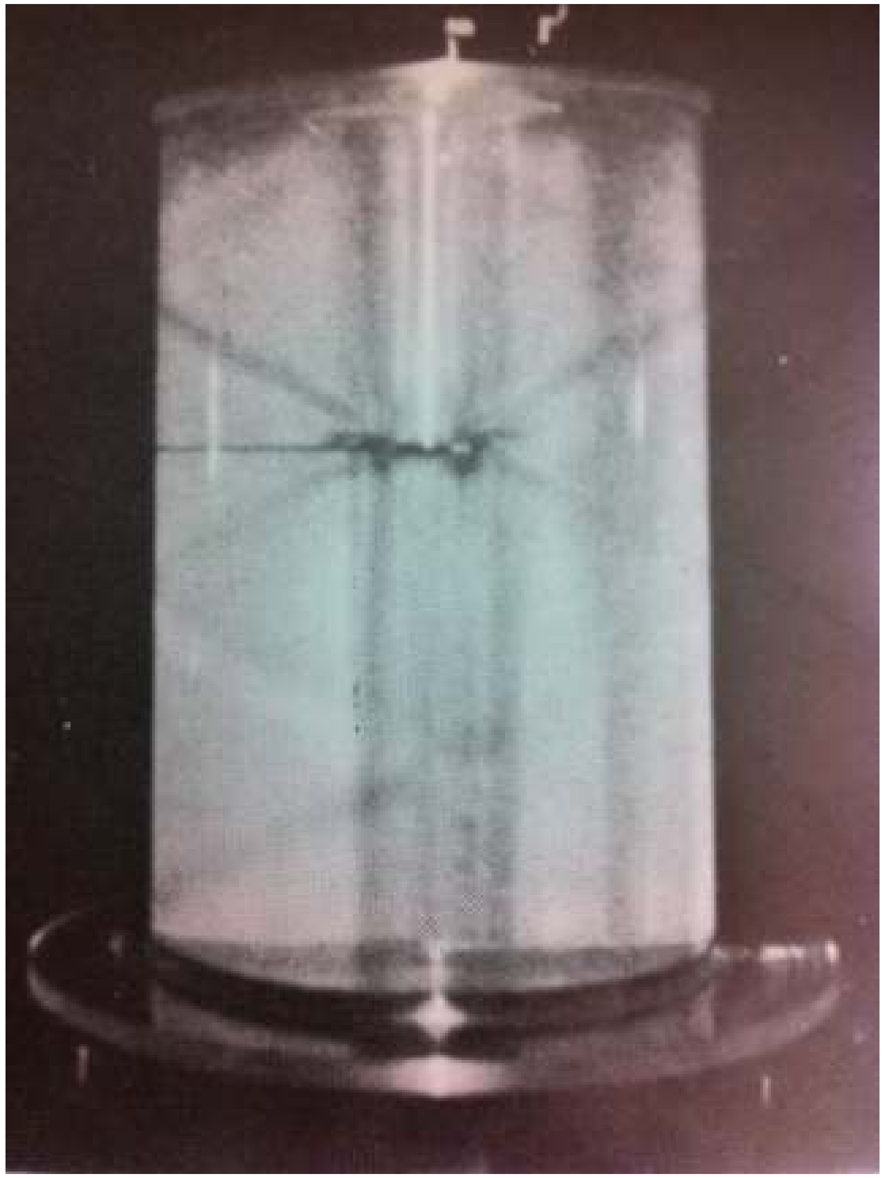
# Reflection of inertial waves:



# Inertial waves in the laboratory

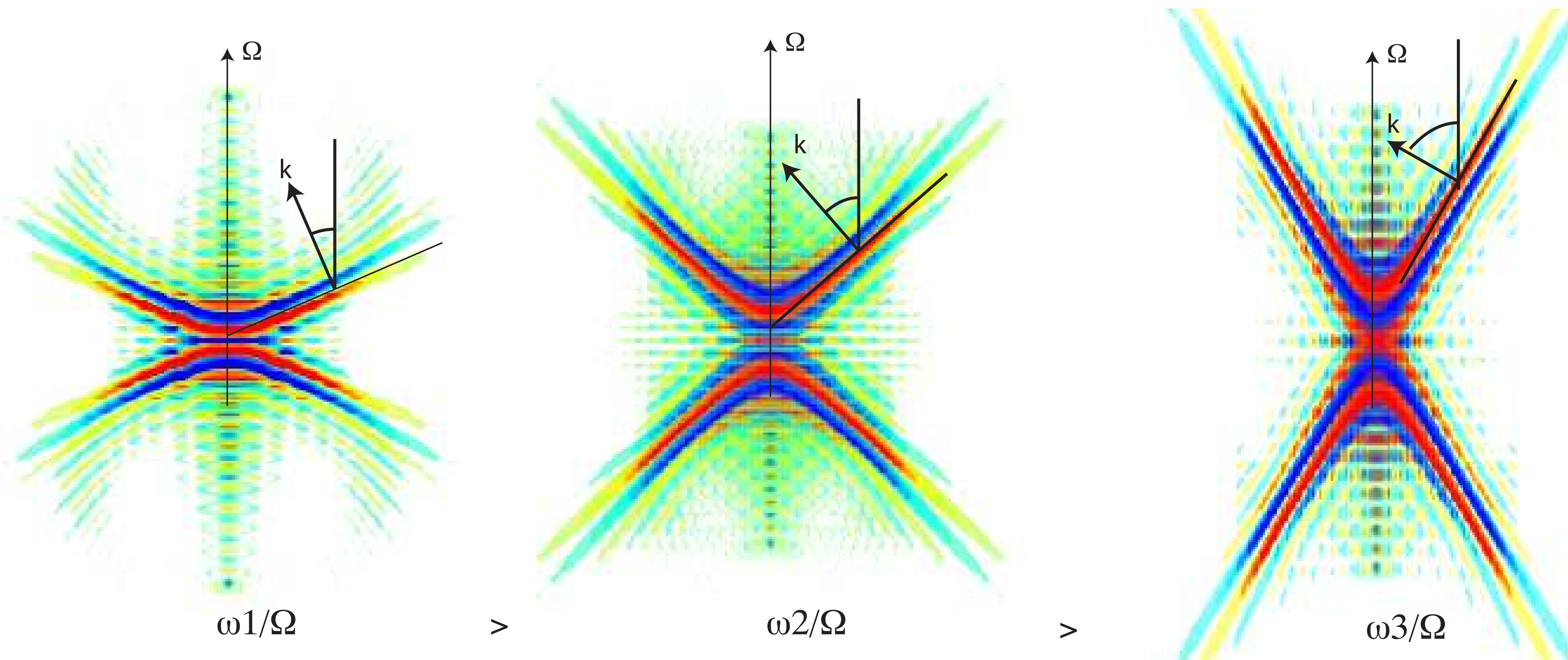



Adapted from a figure by Benjamin Favier, ENS Lyon, France. now at IRPHE, Marseille.



Experiment by Goertler 1957, picture from: The theory of rotating fluids by H. P. Greenspan 1968

# Oscillatory rotating flows.



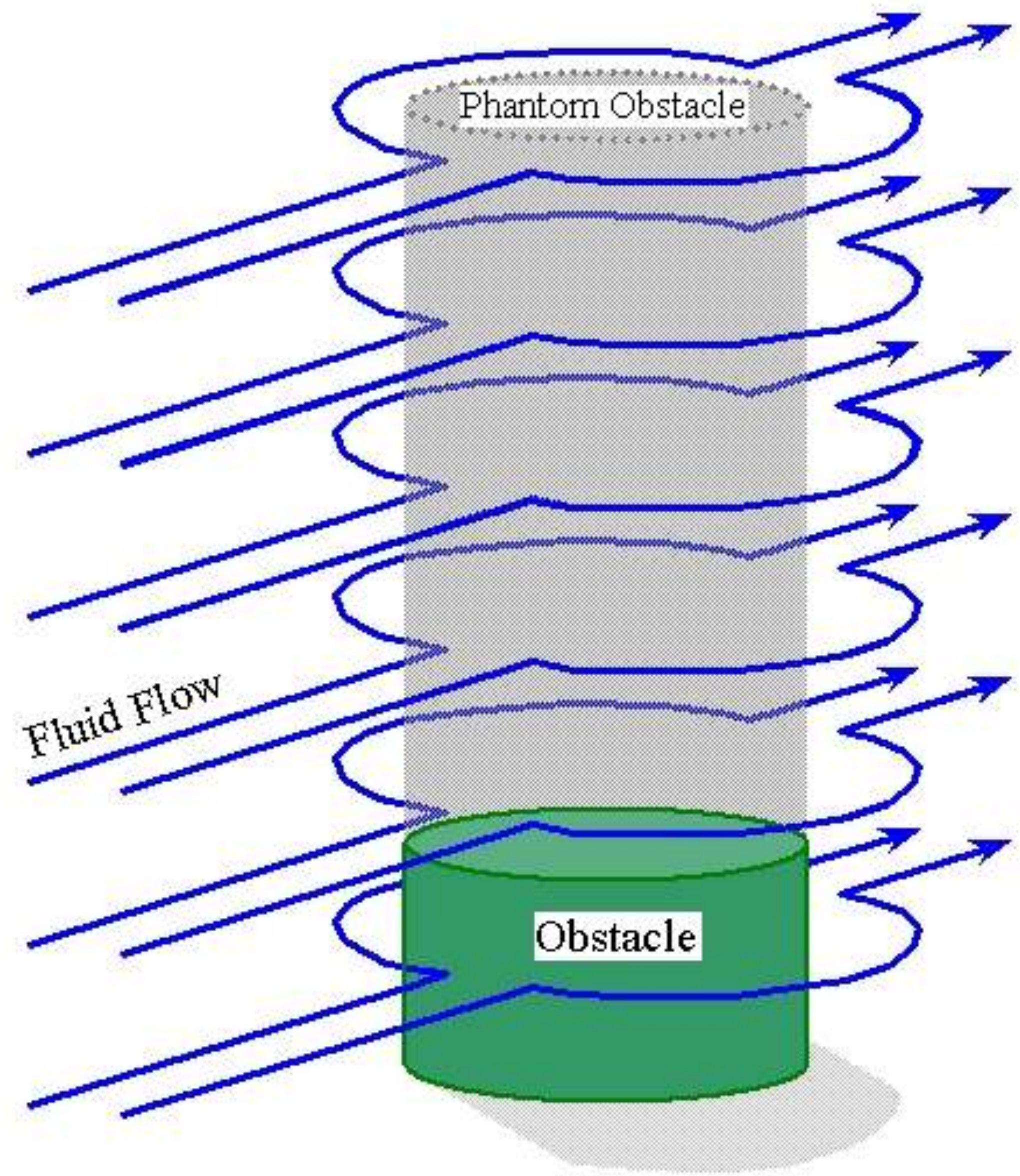

Adapted from a figure by Benjamin Favier, ENS Lyon, France. In agreement with the dispersion relation, the apex angle decreases with the frequency and the distance from the perturbation the energy propagates over one cycle increases. now at IRPHE, Marseille.

# The limit of inertial waves with $\omega$ tends to zero.

- According to the inertial waves properties we just saw, the limit cases $\omega = 0$ would correspond to perturbation that propagates along cones that are degenerated into columns aligned with the axis of rotation.

- Taylor-Proudman is the limit case of the advection of a perturbation by a quasi steady inertial waves.
- The energy propagates parallel to the axis of rotation due the vertical motion.

$$|V_g| = 2\frac{\Omega}{|\mathbf{k}|}$$




Even for a steady perturbation the group velocity remains finite

## Part II:Inertial Waves and Inertial Modes

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# Inviscid Inertial Modes

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi$$

$$\vec{\nabla} \cdot \vec{u} = 0, \quad \hat{\mathbf{n}} \cdot \vec{u} = 0|_S$$

$$\vec{u}_N(t, \vec{r}) = \vec{u}_N e^{i\omega_N t}$$

$$\Pi_N(t, \vec{r}) = \pi_N e^{i\omega_N t}$$

In a closed container the inertial waves equation becomes an eigenvalue problem, the eigenfunctions are the inertial modes.

$$i\omega_N \vec{u}_N + 2\vec{\Omega} \times \vec{u}_N + \vec{\nabla}\Pi_N = 0$$

$$\vec{\nabla} \cdot \vec{u}_N = 0, \quad \hat{\mathbf{n}} \cdot \vec{u}_N = 0|_S$$

- The spectrum is dense.
- They are orthogonal to each other. $\int_V \vec{u}_N \cdot \vec{u}_M dv = 0$
- In certain well behaved geometries, they form a base.

$$\vec{u} = \sum_n a_n \vec{u}_n e^{i\omega_n t}$$

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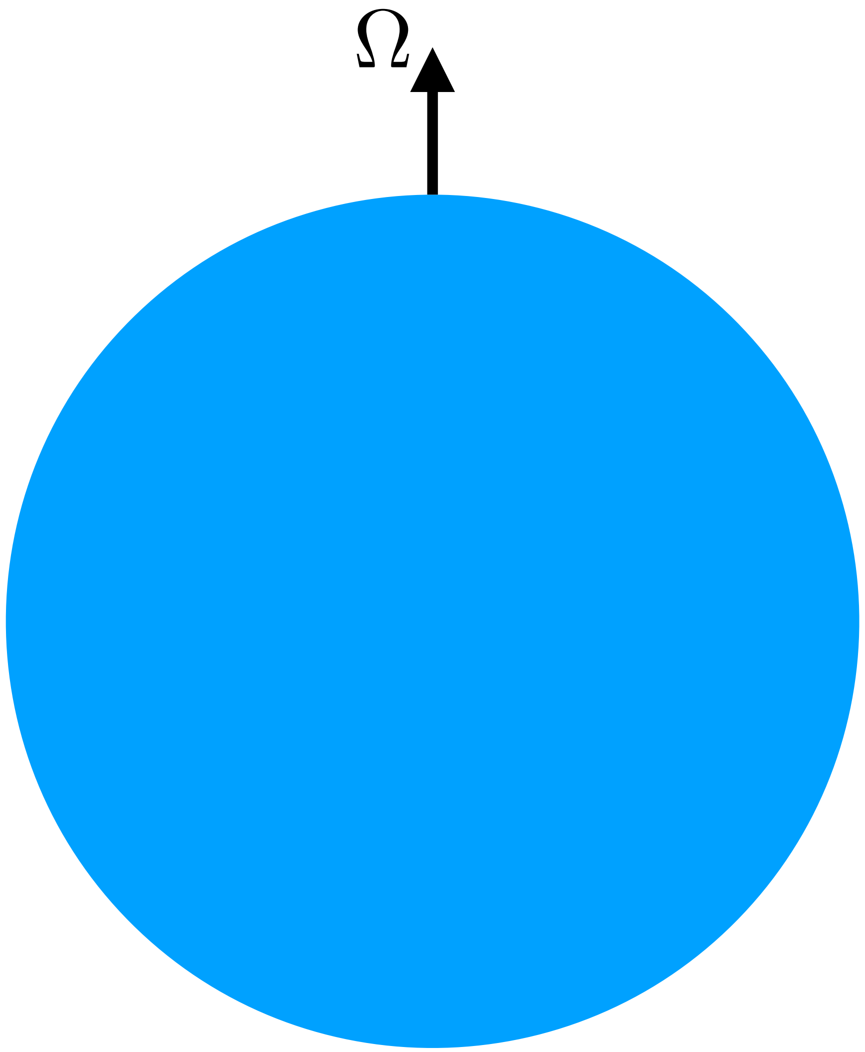
# First order viscous correction to the inviscid inertial modes

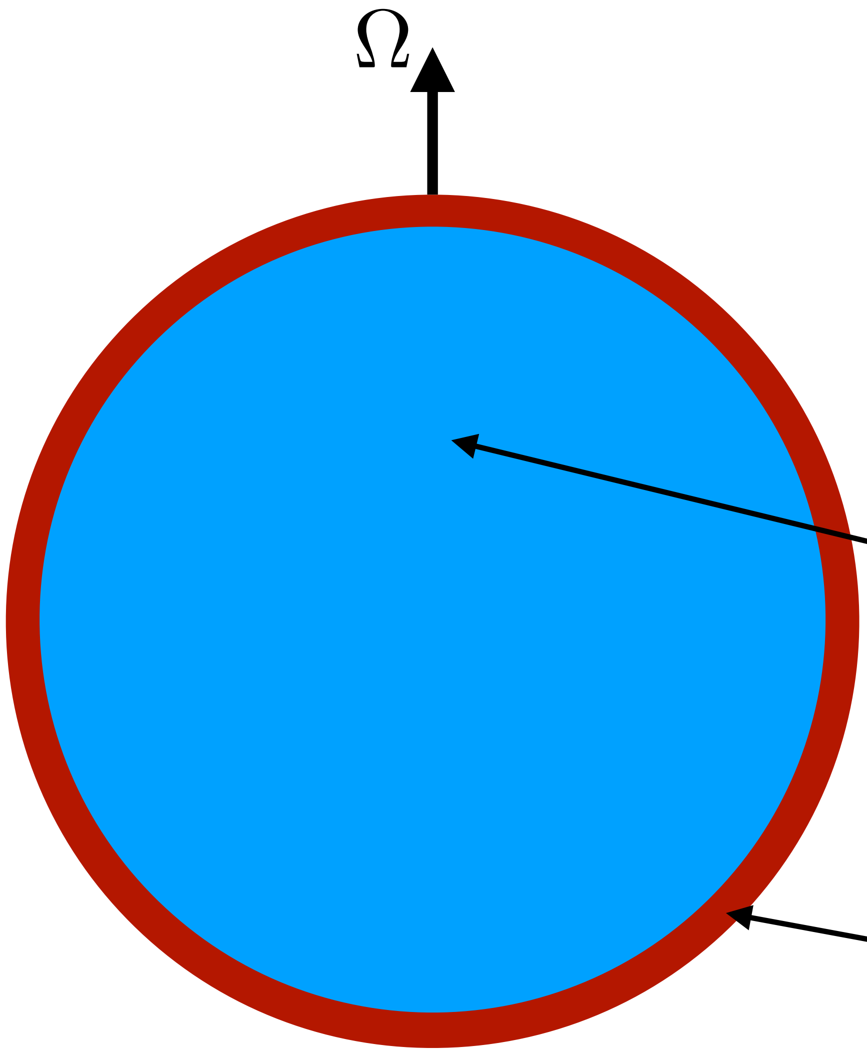



$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi + E\vec{\nabla}^2\vec{u} + \vec{r}$$

$$\vec{\nabla} \cdot \vec{u} = 0, \quad \boxed{\vec{u} = 0|_S}$$

**Inviscid inertial modes do not satisfy the no-slip boundary conditions.**

# First order viscous correction to the inviscid inertial modes




$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi + E\vec{\nabla}^2\vec{u}$$

$$\vec{\nabla} \cdot \vec{u} = 0, \quad \vec{u} = 0|_S$$

**Inviscid inertial modes do not satisfy the no-slip boundary conditions.**

**Inviscid interior at large scale** $\quad 2\vec{\Omega} \times \vec{u} >> E\vec{\nabla}^2\vec{u}$

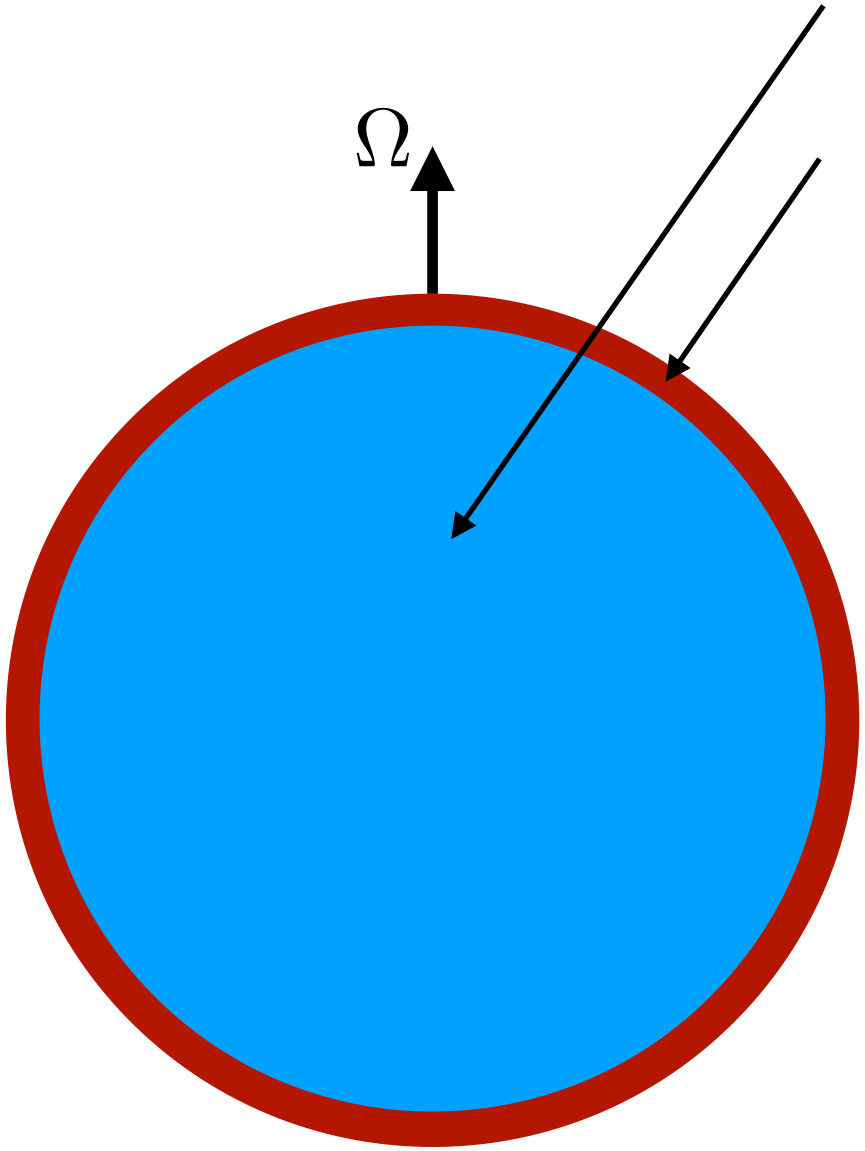
$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi$$

**Viscous Boundary Layer.** $\quad 2\vec{\Omega} \times \vec{u} \sim E\vec{\nabla}^2\vec{u}$

$\delta \sim \sqrt{E}$ $\qquad \sim \mathcal{O}(1)$

# First order viscous correction to the inviscid inertial modes

**Following Greenspan 1968:**




$$\vec{u} = \vec{u}_N^0 + \sqrt{E}\vec{u}_N^1 + \mathcal{O}(E^\alpha)$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_N^0 + \sqrt{E}\tilde{\mathbf{u}}_N^1 + \mathcal{O}(E^\alpha)$$

$$\omega_N = \omega_N^0 + i\lambda_N\sqrt{E} + \mathcal{O}(E^\alpha), \alpha > \frac{1}{2}$$

**1-Inviscid interior at leading order**

$$\frac{\partial \vec{u}_0}{\partial t} + 2\vec{\Omega} \times \vec{u}_0 = -\vec{\nabla}\Pi_0 \qquad \hat{\mathbf{n}} \cdot \vec{u}_0|_S = 0$$

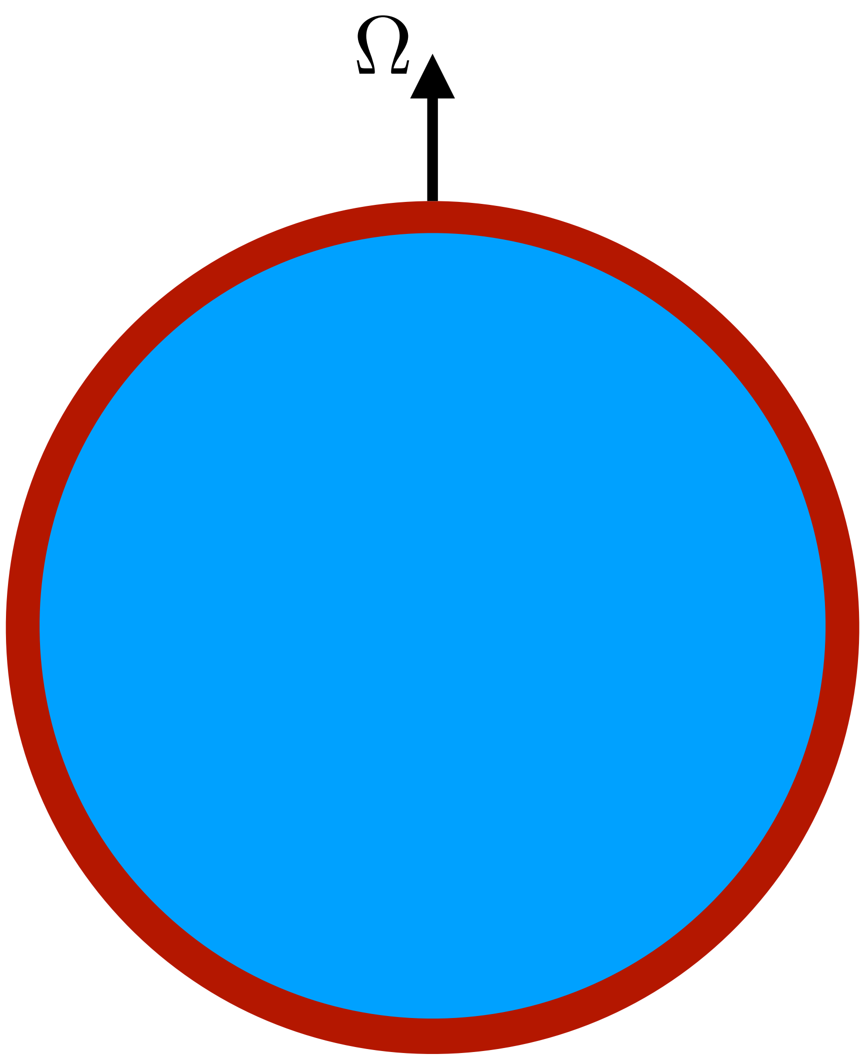
**2-Viscous Boundary Layer at leading order.**

$$\frac{\partial \tilde{\vec{u}}}{\partial t} + 2\vec{\Omega} \times \tilde{u} = -\vec{\nabla}\tilde{\Pi} + E\vec{\nabla}^2\tilde{u} \qquad \vec{u}_0 + \tilde{\mathbf{u}}_0|_S = 0$$

**3-Inviscid interior at order** $\sqrt{E}$ $\qquad \hat{\mathbf{n}} \cdot (\vec{u}_1 + \tilde{\mathbf{u}}_1) = 0$

The solvability condition lead to an expression of $\lambda_N$ , the rate at which an inertial mode N will decay in time if not forced continuously.

# First order viscous correction to the inviscid inertial modes



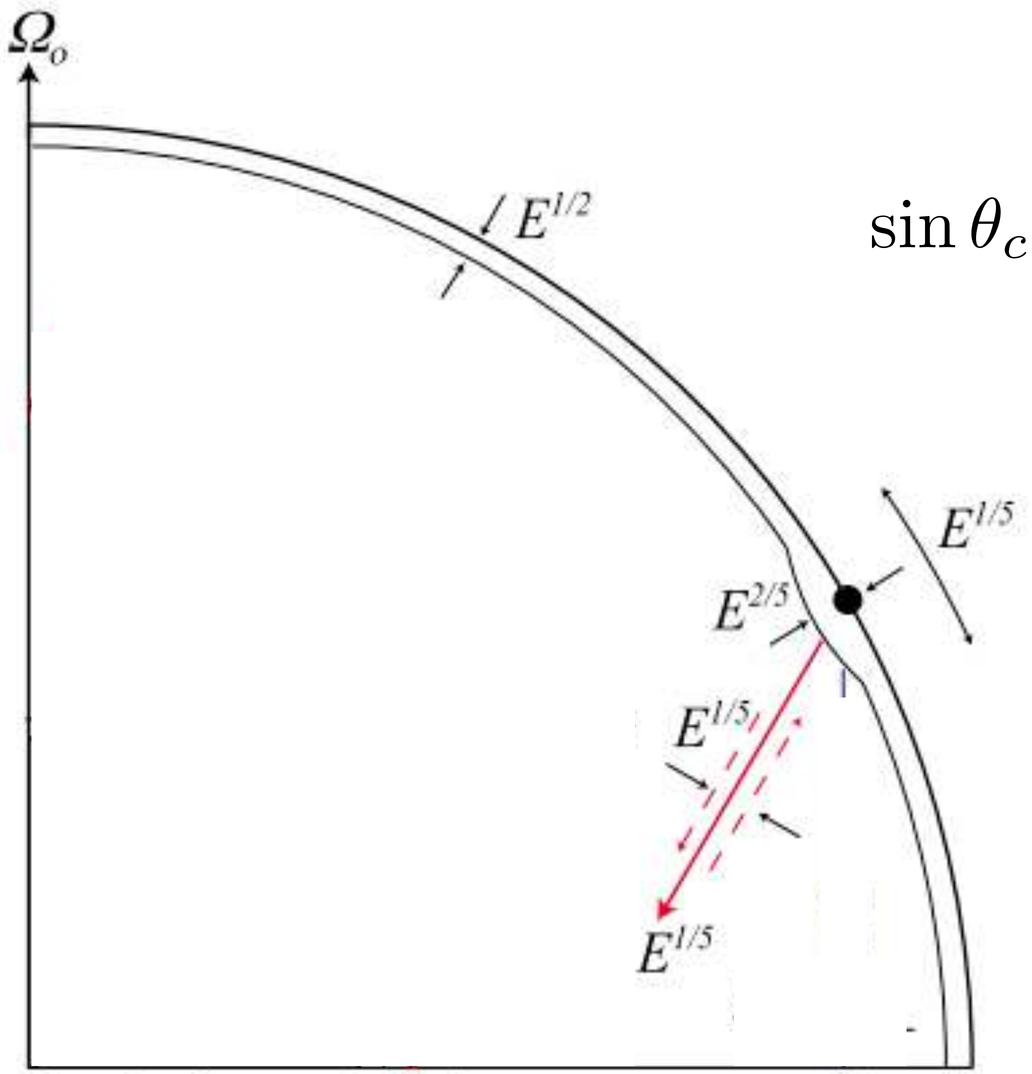

**Summary:**

Inertial modes with viscous correction have at leading order the **structure of the inviscid mode** on which superimposed a small flux coming from the Ekman boundary layer, the so-called **Ekman pumping**. By friction in the boundary layer, they dissipate energy, this viscous damping is characterised by a **decay rate.**

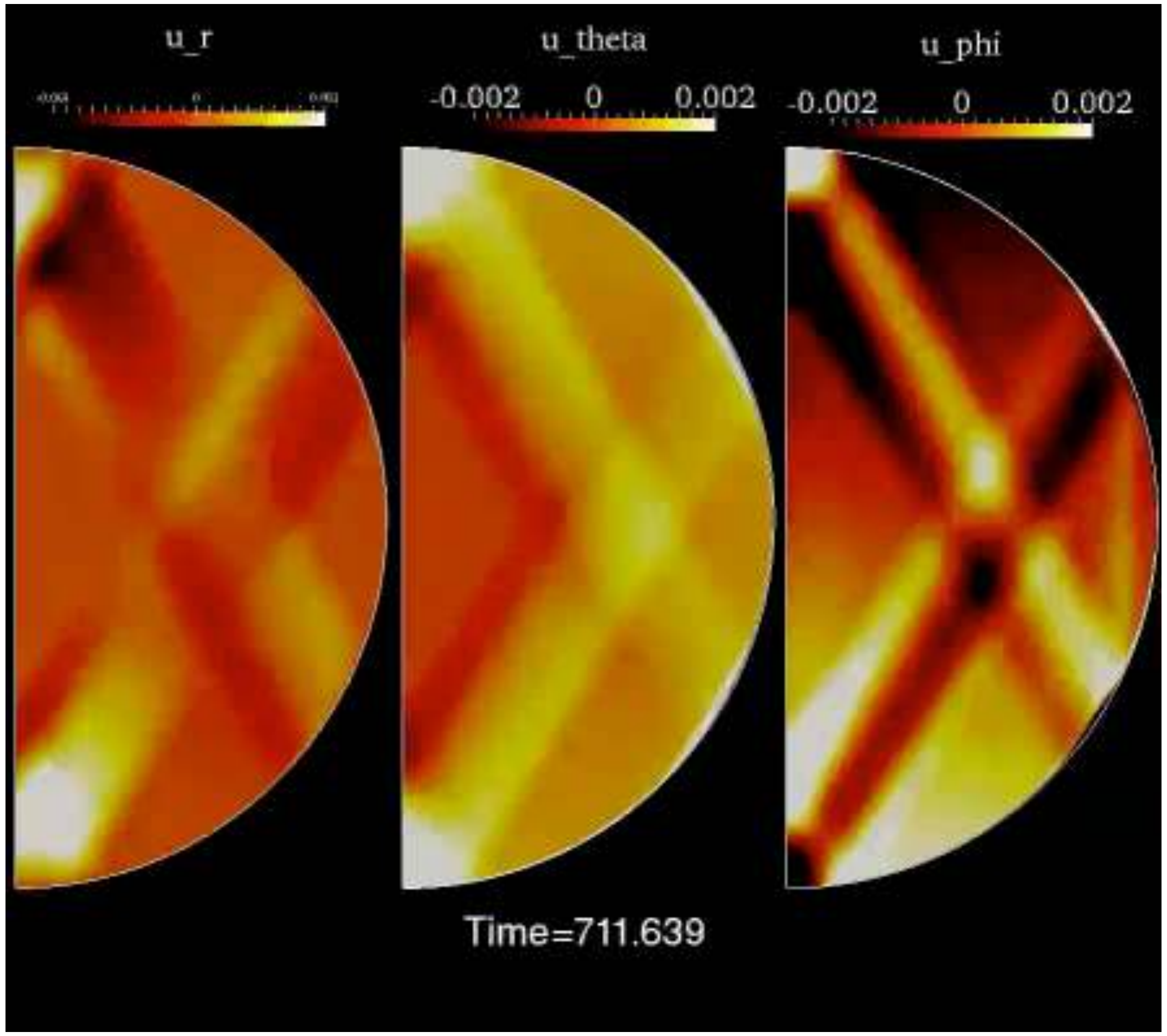
Further corrections can be made by pushing the expansion to the next order in E.

The secondary flow driven by the Ekman pumping: What do you see if you rotate with the fluid ?

Ekman pumping driven inertial waves




$$\sin\theta_c = \frac{\omega}{2\Omega}$$

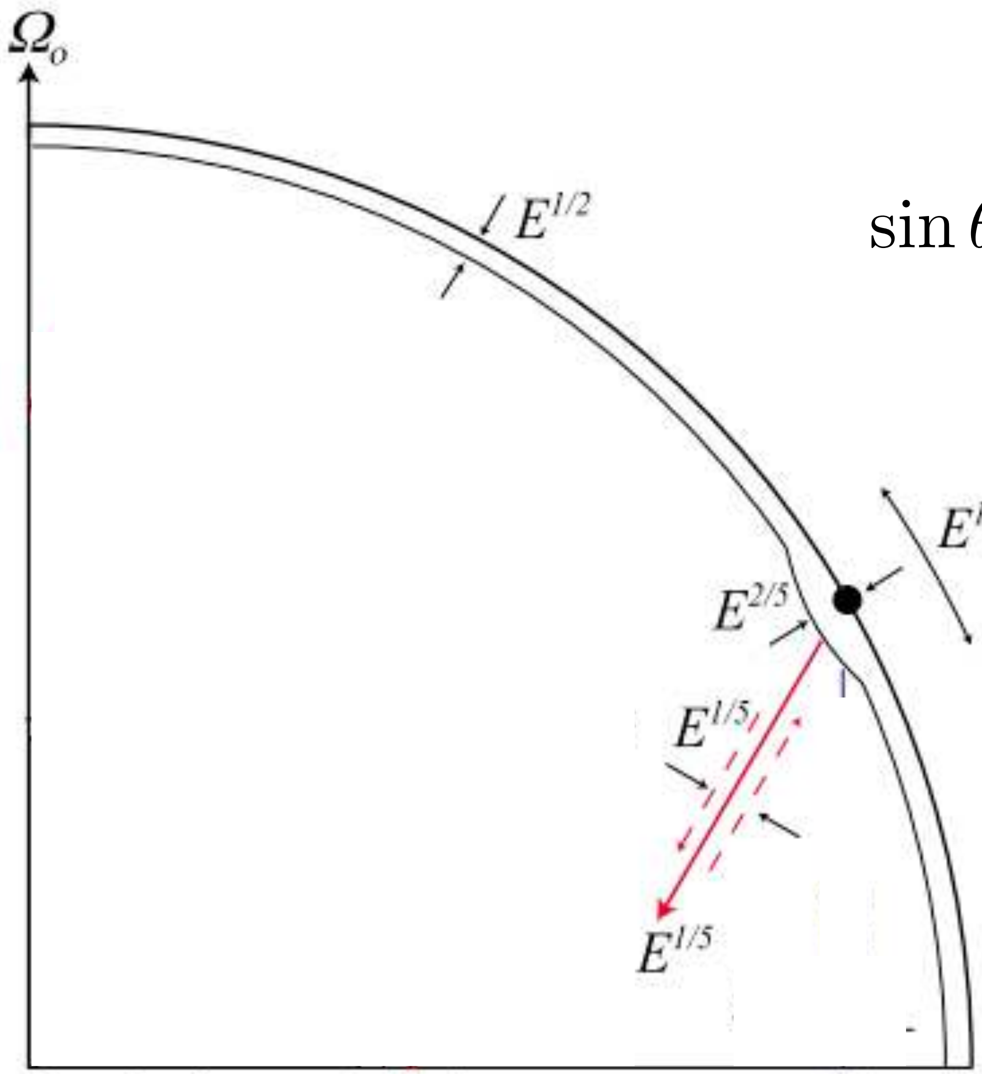



courtesy of Yufeng Lin

Bondi and Littleton, Stewartson and Roberts 1963, Kerswell 1995, Noir et al. 2001, Kida 2011

The secondary flow driven by the Ekman pumping: What do you see if you rotate with the fluid ?

Ekman pumping driven inertial waves

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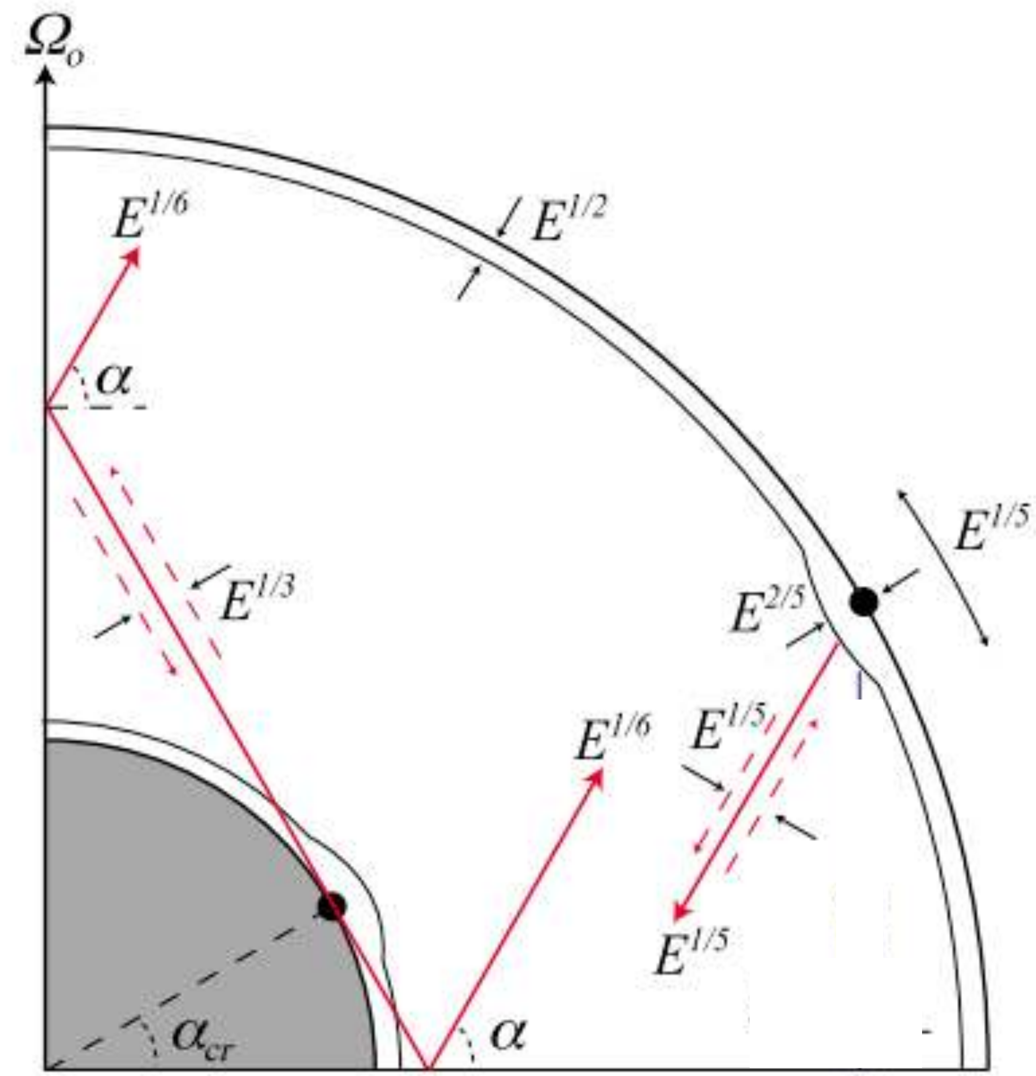
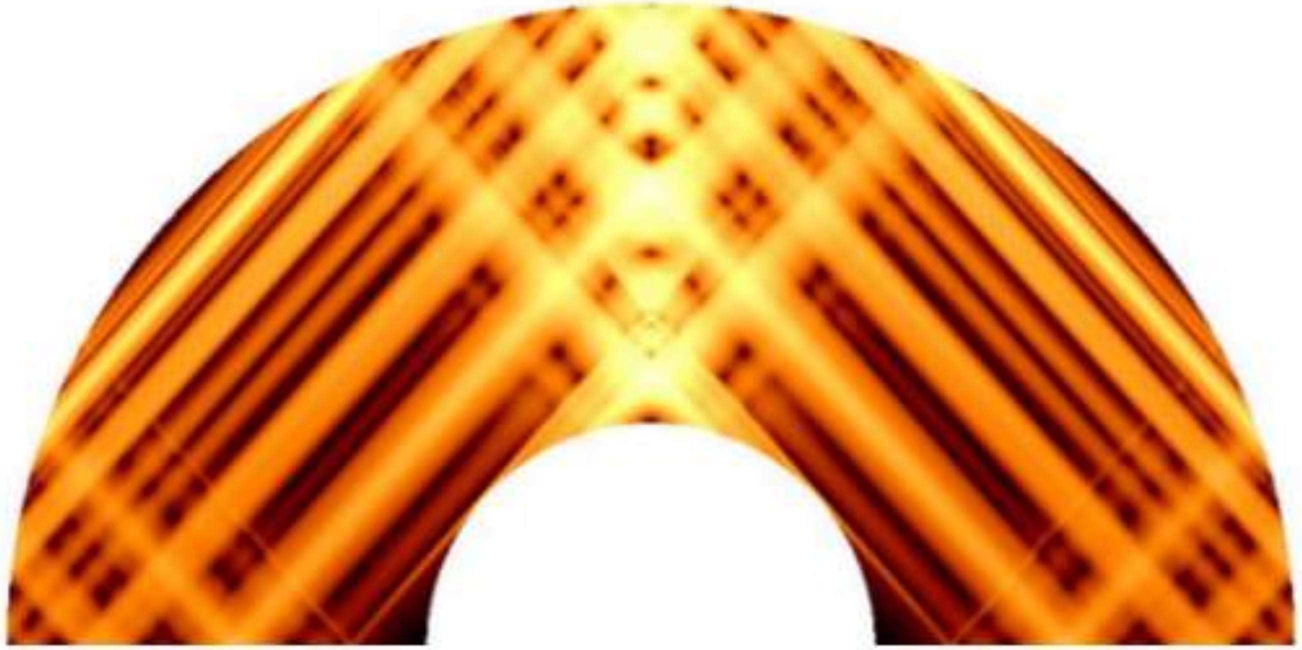
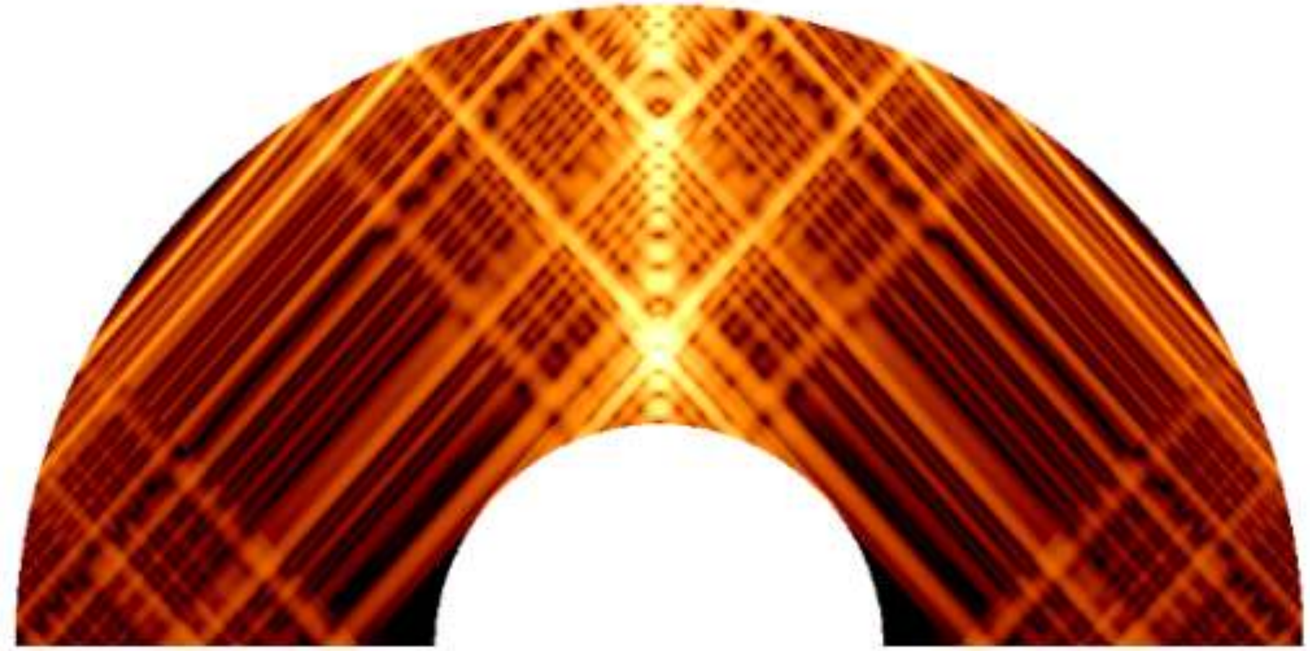


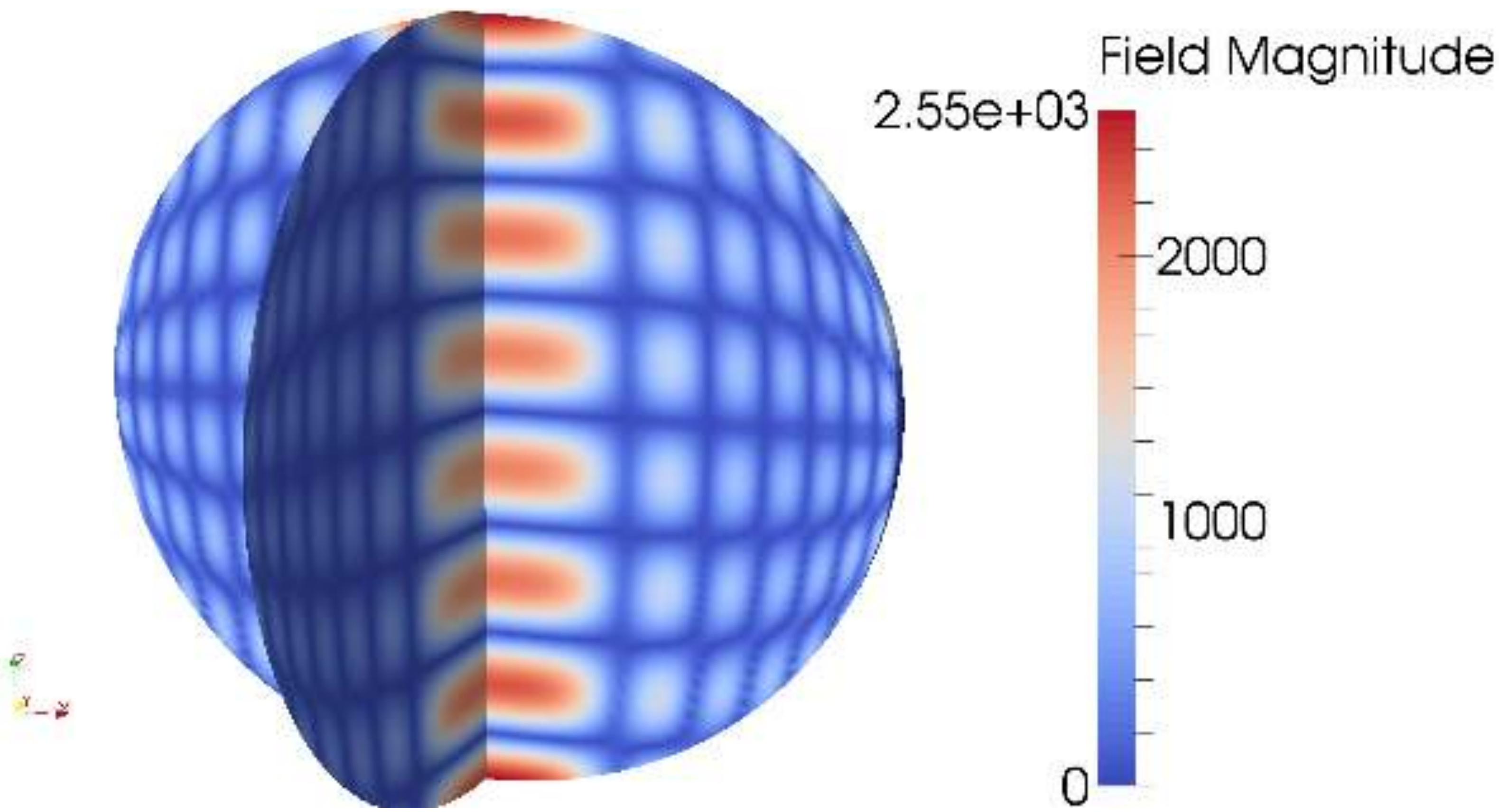

Bondi and Littleton, Stewartson and Roberts 1963, Kerswell 1995, Noir et al. 2001, Kida 2011



$\omega_i/\Omega_s = 0.66277$



$\omega_i/\Omega_s = 0.6597$

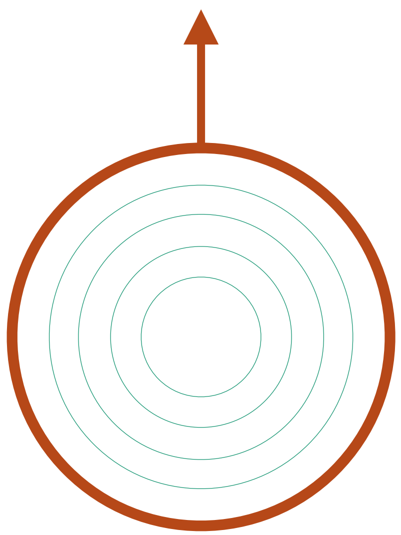



**Courtesy of Jeremie Vidal (Grenoble/ Leeds)**

# The Poincaré mode in spheroidal shell: The simplest inertial mode.

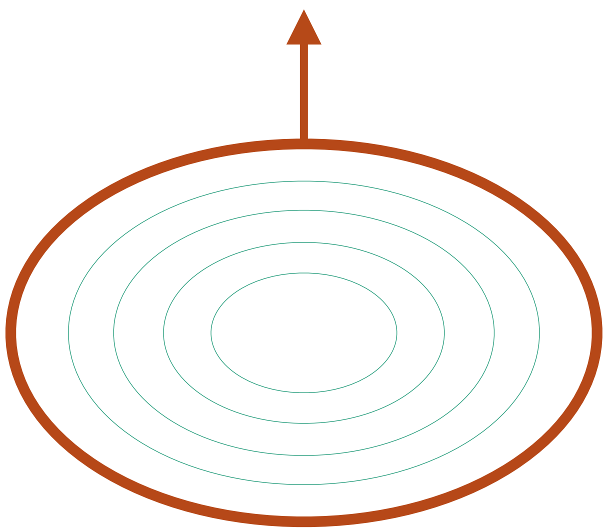
The simplest inertial mode is a quasi solid body rotation along an equatorial direction.known under various names:

- Q211 mode, Spin-over mode. (Greenspan)
- Poincaré mode. (Too many people)
- Tilt-over mode, (Toomre, Noir, Zhang)
- (110)-mode (K. Zhang)
- FCN: Free Core Nutation (Astronomers)



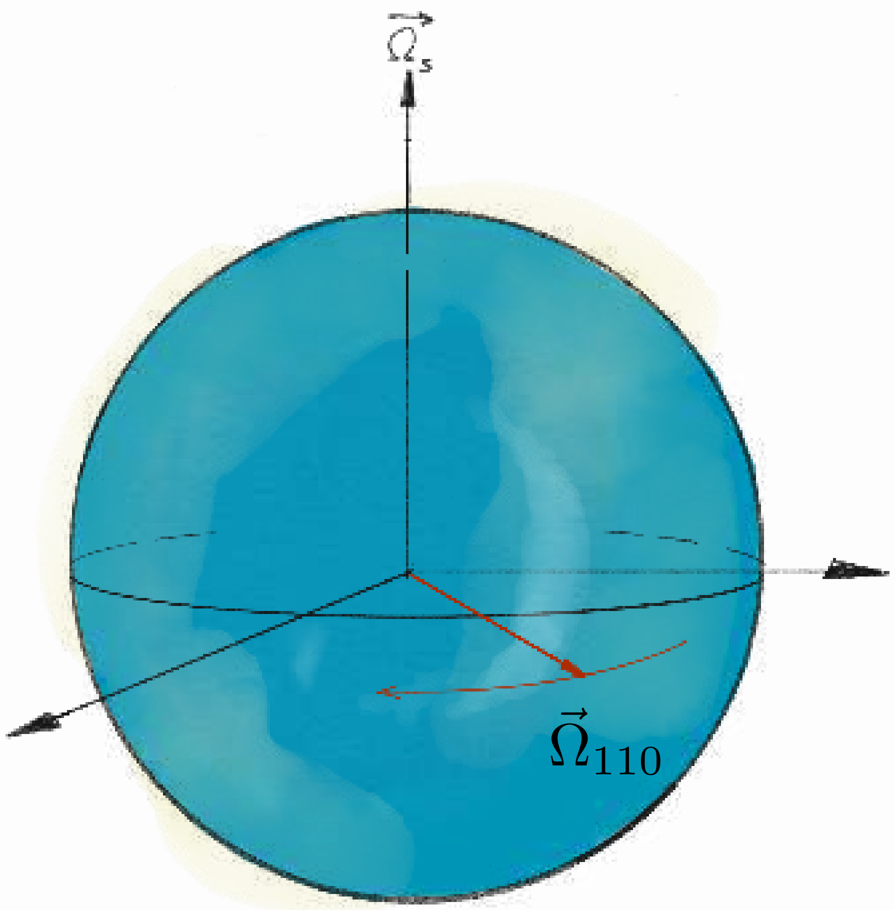
**Sphere** = purely circular streamlines

$$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r}$$



**Spheroid** = elliptical streamlines

$$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r} + \vec{\nabla}\phi$$




$$\frac{\omega_{110}}{\Omega} = \frac{2}{1+(1-\eta)^2}$$

**This mode has a uniform vorticity, the velocity is linear is the spatial coordinates.**

## Part II:Inertial Waves and Inertial Modes

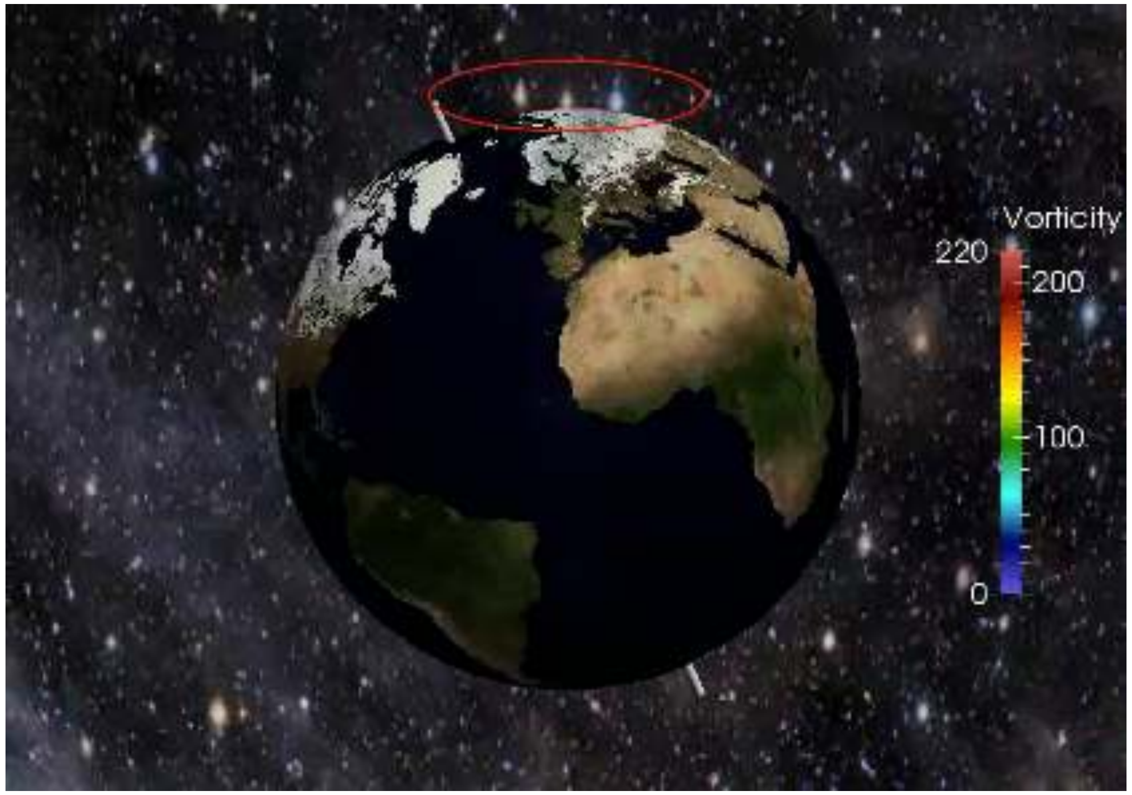
1. Some observations.
2. Inviscid Inertial Waves.
3. Inviscid Inertial Modes.
4. Viscous Correction.
5. Resonances, large flows from small perturbations.

## Probing planetary interiors through the variation of their rotation.

Librations

nasa.gov

Precession / Nutations

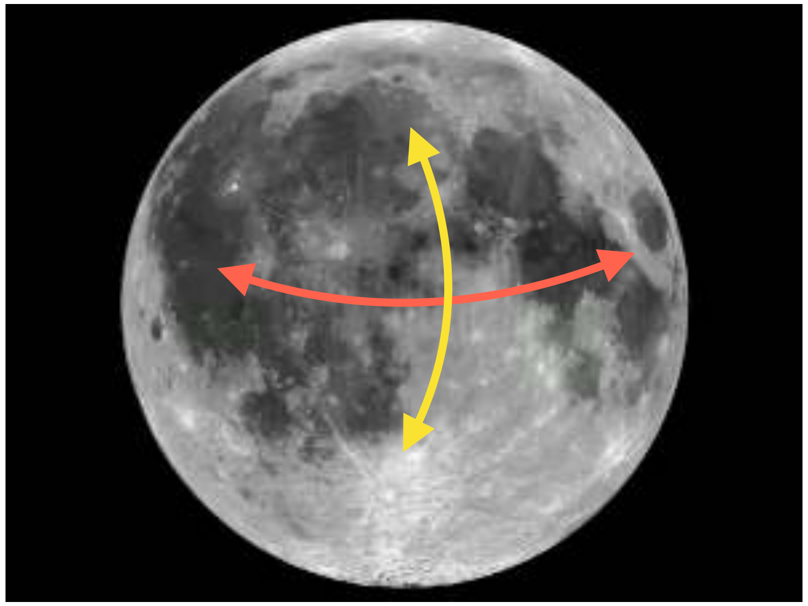


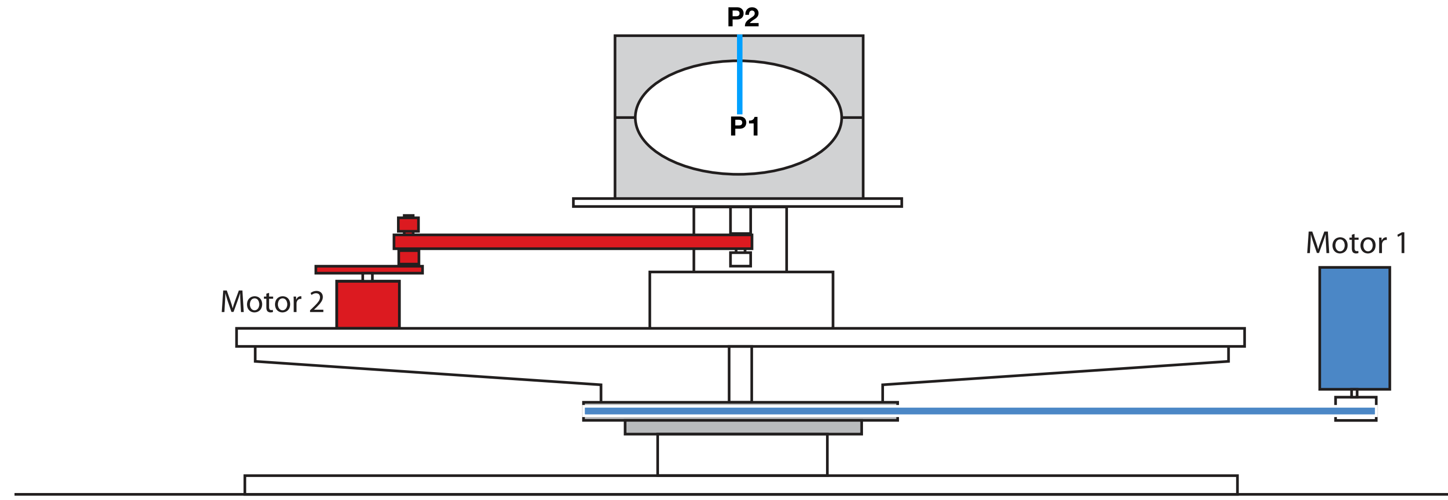

nasa.gov

Tides

nasa.gov

# Libration in Longitude






**Axisymmetric inertial oscillations of a fluid in a rotating spherical container**


By KEITH D. ALDRIDGE† AND ALAR TOOMRE

Massachusetts Institute of Technology, Cambridge, Massachusetts



(Received 22 August 1968)

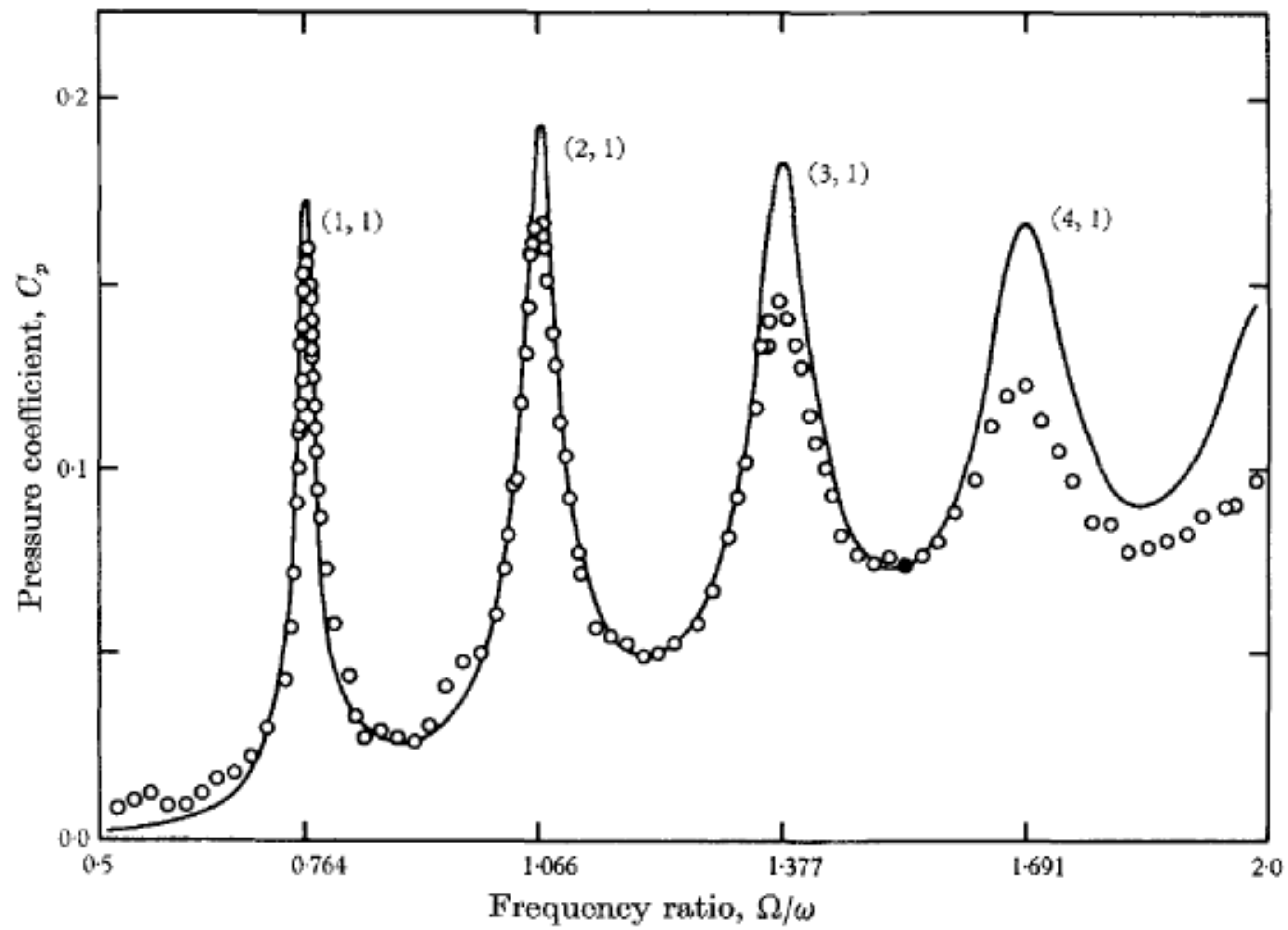
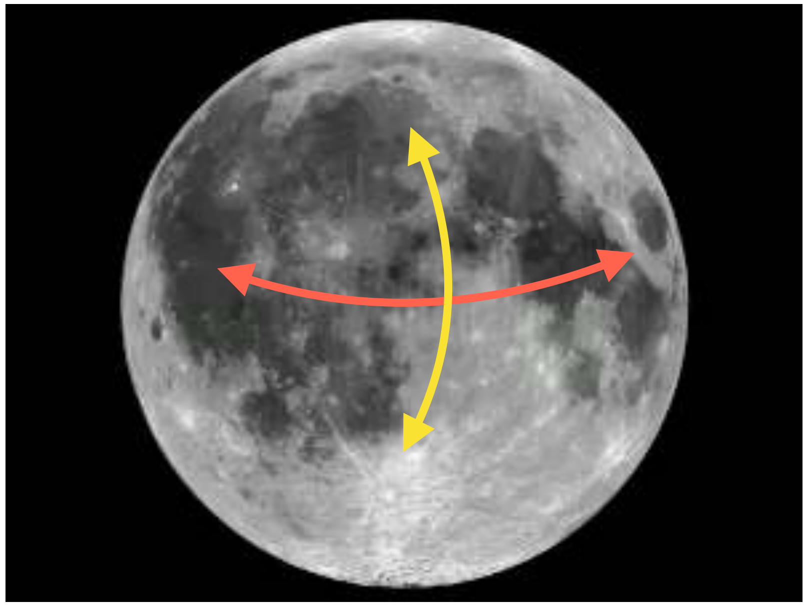




FIGURE 3. Pressure amplitudes at the centre of the sphere, for $\epsilon = 8{\cdot}0°$.

# Libration in Latitude



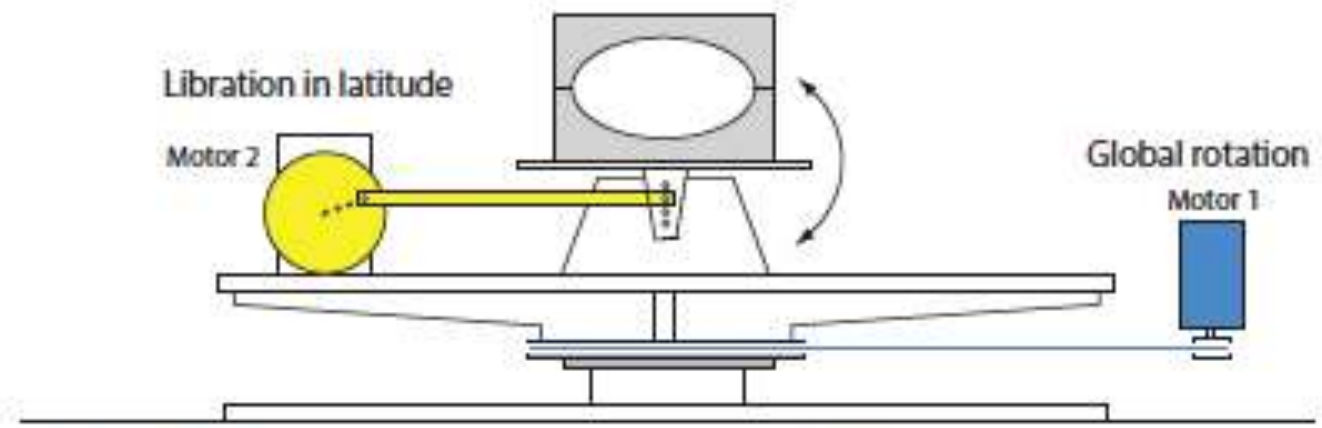
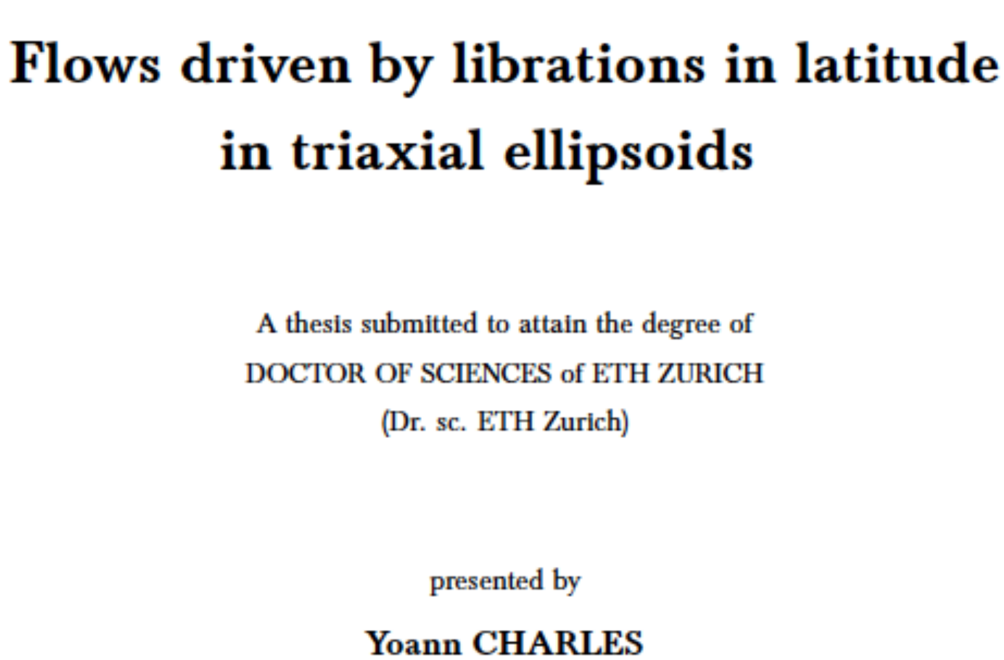
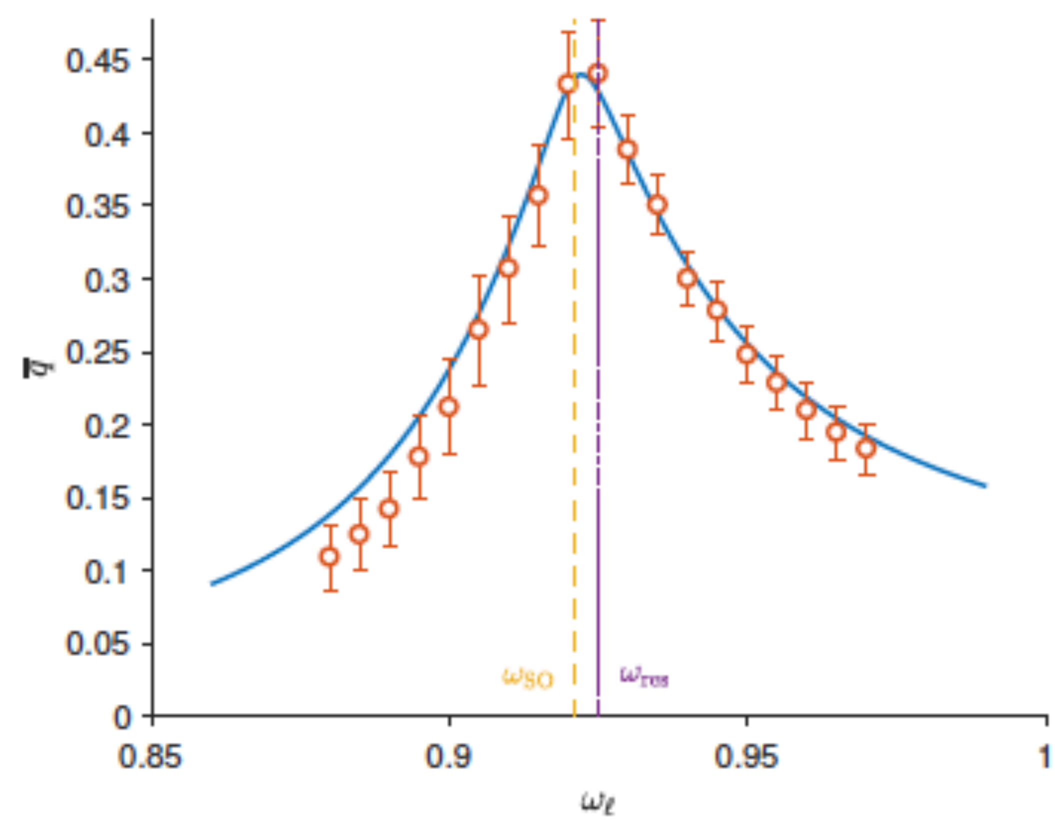



Figure 2.1 Schematic representation of a latitudinal libration experiment.

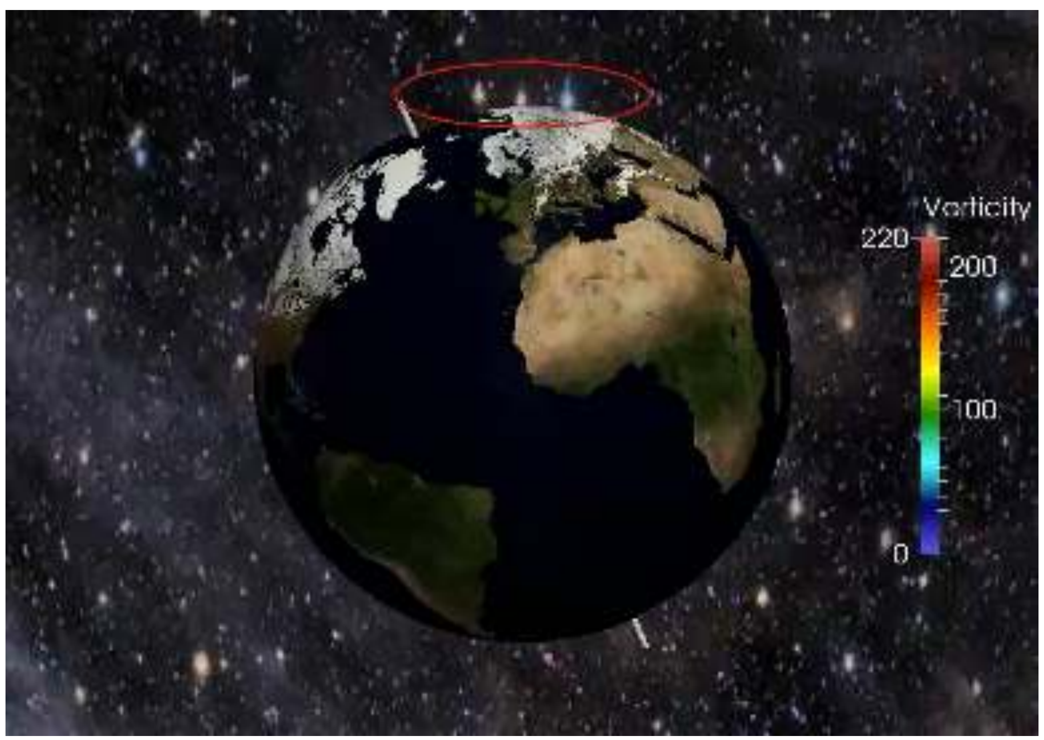
Fluid rotation amplitude from the point of view of an observer located on the mantle





# Planetary core flows driven by precession:

Let's consider a homogeneous fluid, of uniform density, enclosed in a rapidly rotating spheroidal cavity.
The cavity shape is characterised by (a,c) the equatorial and polar radius respectively.




Mantle $\vec{\Omega}_s$ $\vec{\Omega}_p$ Precession

Noir 2000

Control parameters

$$E = \frac{\nu}{\Omega_s R^2}$$

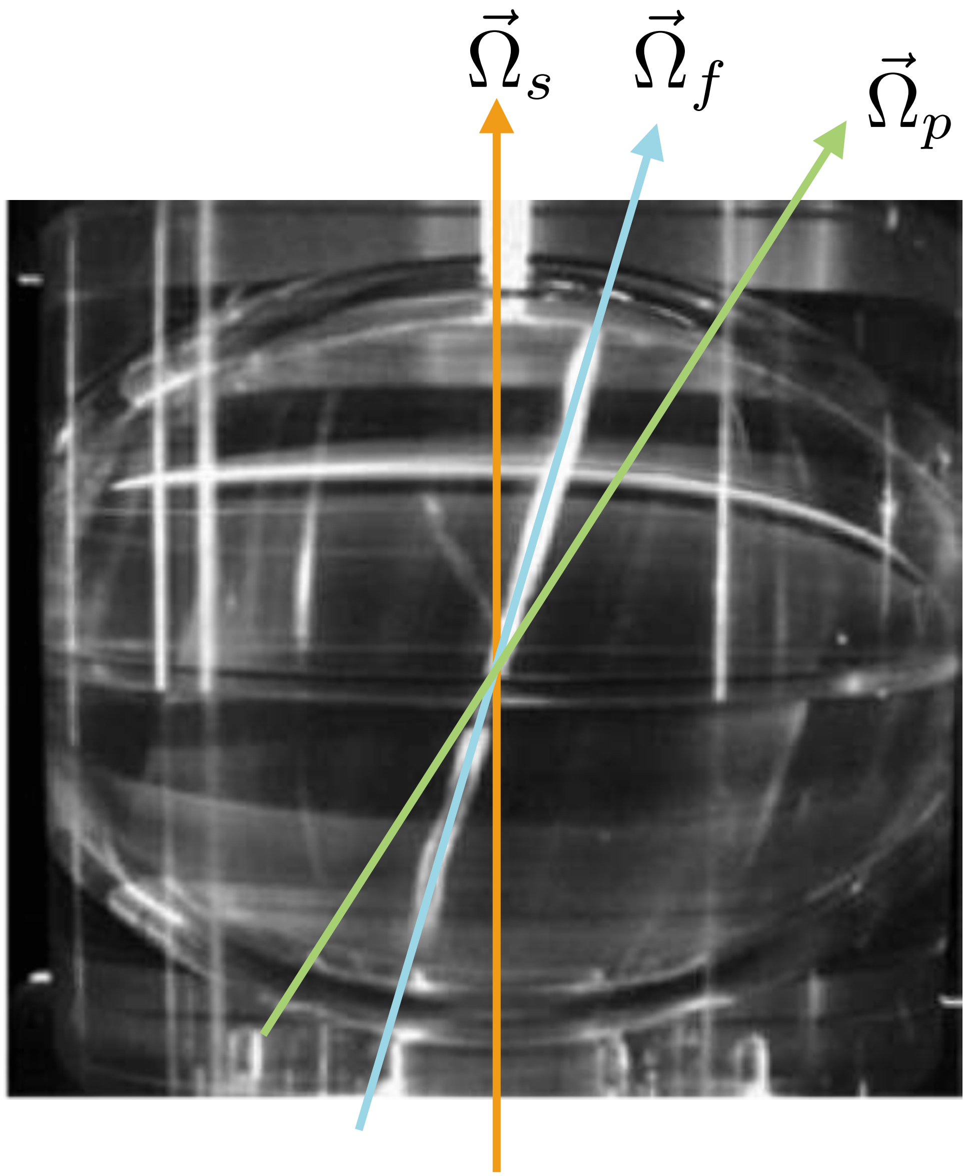
$$Po = \frac{\Omega_p}{\Omega_s}$$

$$\eta = \frac{a-c}{a}$$

$$\alpha$$

# As the mantle precesses, the core remains in rotation but along a slightly tilted axis.

Sloudsky 1895, Poincare 1910, Busse 1968,
Noir and Cebron 2015, Cebron et al 2019.





In the frame of precession, i.e. the turntable, the rotation of the fluid and of the mantle are fixed, viewed from the lab both axis are precessing at the same rate.

Let's look at the equations in the frame attached to the mantle, $\vec{\Omega} = \vec{\Omega}_s + \vec{\Omega}_p$

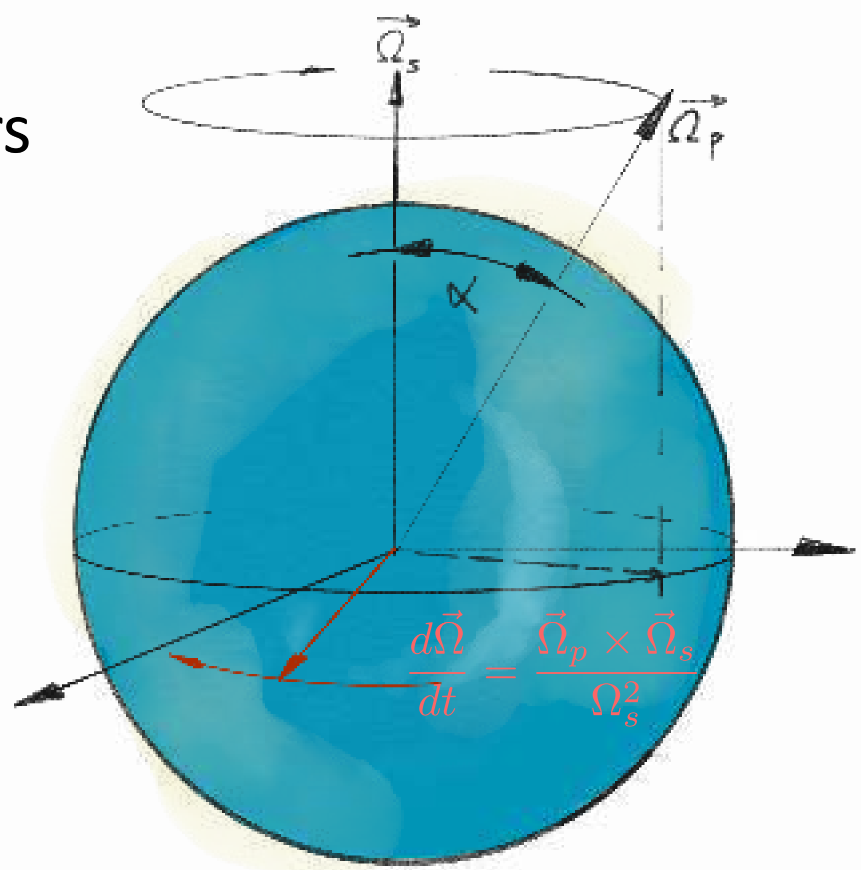
Control parameters

$$\eta = \frac{a-c}{a} \sim 10^{-4}$$

$$E = \frac{\nu}{\Omega_s a^2} \sim 10^{-12}$$

$$Po = \frac{\Omega_p}{\Omega_s} \sim -10^{-4}$$

$$\alpha = 1.5°$$




$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi + E\vec{\nabla}^2\vec{u} + \vec{r} \times \frac{d\vec{\Omega}}{dt}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$\vec{u} = 0$ At the

$$\frac{d\vec{\Omega}}{dt} = \frac{\vec{\Omega}_p \times \vec{\Omega}_s}{\Omega_s^2}$$

Viewed from an observer fixed on the mantle, this vector travels in a retrograde direction with the period of rotation of the mantle:

The Poincare acceleration takes the form of a solid body rotation and appears as a forcing term in the equations.

Let’s look at the equations in the frame attached to the mantle, $\vec{\Omega} = \vec{\Omega}_s + \vec{\Omega}_p$

## Control parameters

$$\eta = \frac{a-c}{a} \sim 10^{-4}$$

$$E = \frac{\nu}{\Omega_s a^2} \sim 10^{-12}$$

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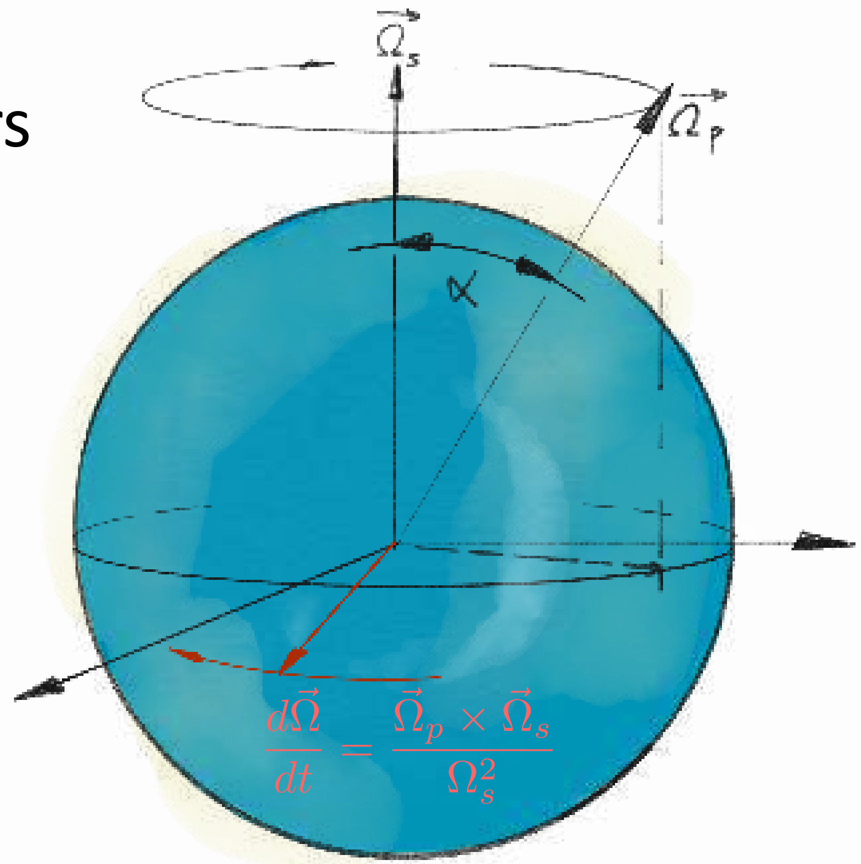


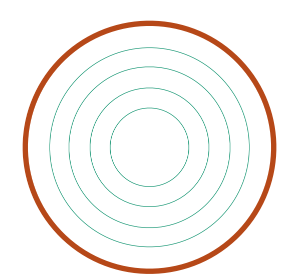

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi + \vec{r} \times \frac{d\vec{\Omega}}{dt}$$

$$\frac{d\vec{\Omega}}{dt} = Posin\alpha(\cos(t)\hat{\mathbf{e}}_x + \sin(t)\hat{\mathbf{e}}_y)$$

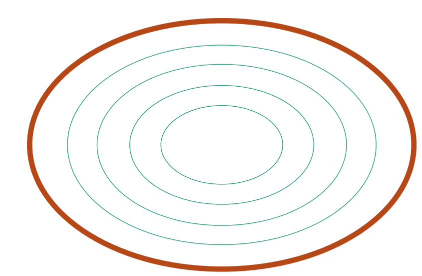
$$\vec{\nabla} \cdot \vec{u} = 0$$

## Poincaré mode:

$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r}$

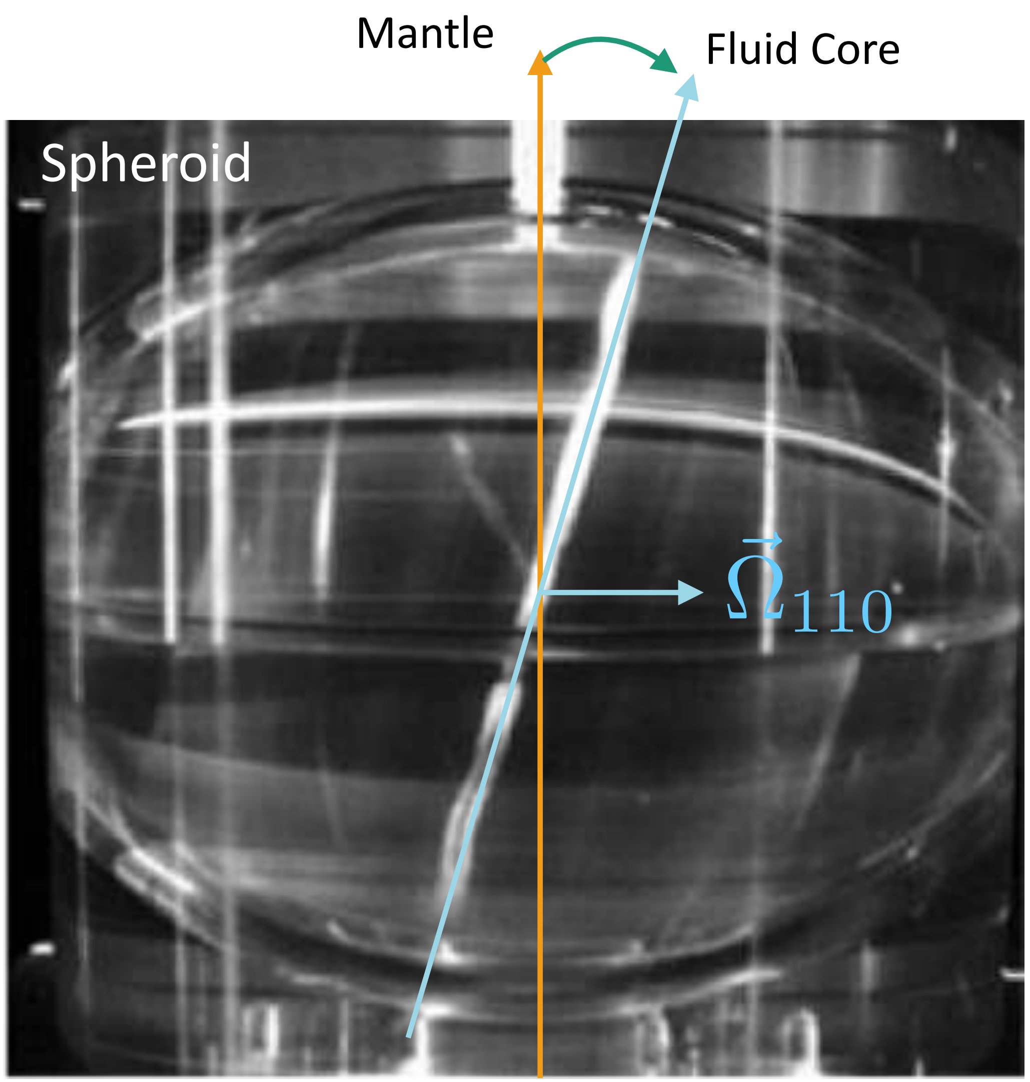


$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r} + \vec{\nabla}\phi$



The Poincare acceleration takes the form of a solid body rotation and appears as a forcing term in the equations.

Let’s look at the equations in the frame attached to the mantle, $\vec{\Omega} = \vec{\Omega}_s + \vec{\Omega}_p$



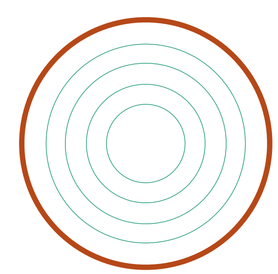

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla}\Pi + \vec{r} \times \frac{d\vec{\Omega}}{dt}$$

$$\frac{d\vec{\Omega}}{dt} = Posin\alpha(\cos(t)\hat{\mathbf{e}}_x + \sin(t)\hat{\mathbf{e}}_y)$$

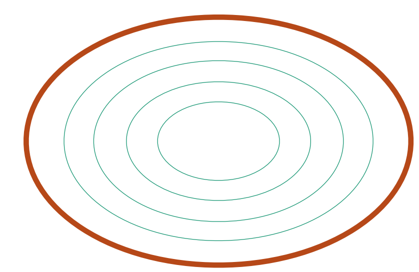
$$\vec{\nabla} \cdot \vec{u} = 0$$

Poincaré mode:

$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r}$



$\vec{u}_{110} = \vec{\Omega}_{110} \times \vec{r} + \vec{\nabla}\phi$

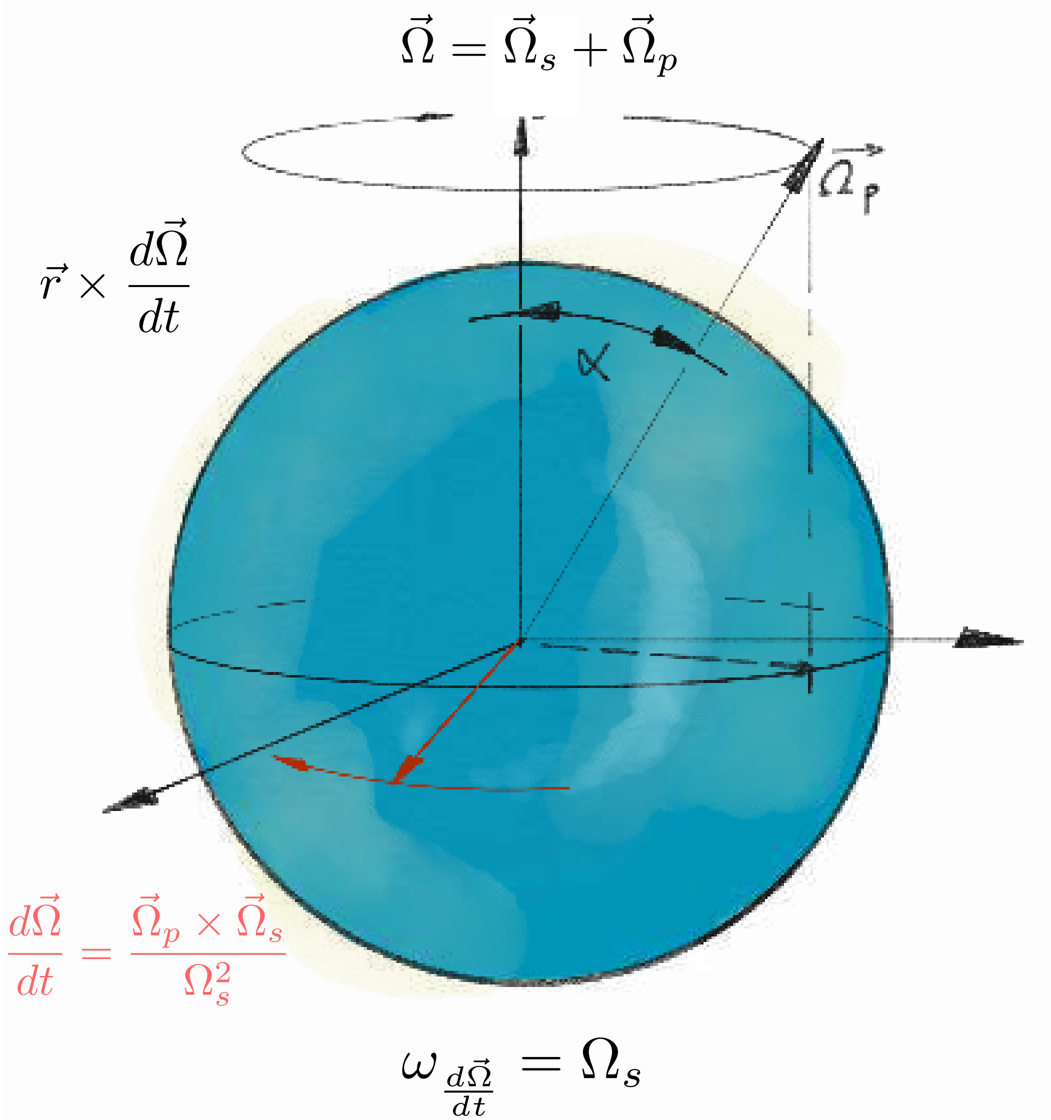


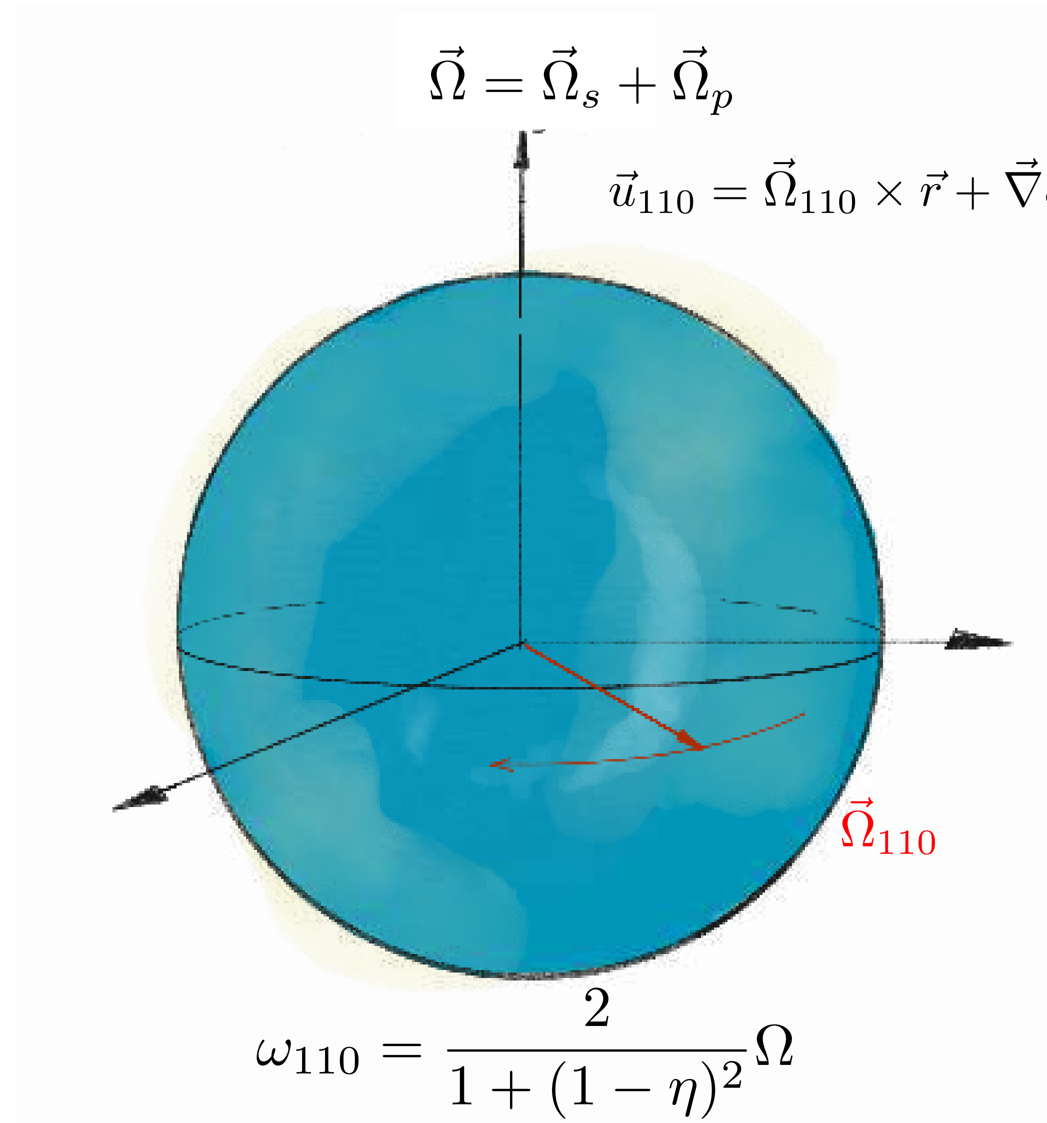
The Poincare acceleration takes the form of a solid body rotation and appears as a forcing term in the equations.

## To go further

1. Greenspan section 2
2. K.K. Zhang section 2
3. S. Vantieghem 2014, PRSL, Inertial modes in a rotating triaxial ellipsoid,
4. Ivers et al. 2015, JFM: Enumeration , orthogonality and completeness of the incompressible Coriolis modes in a sphere.

# Resonance between the Poincaré mode (FCN) and the Precession / Nutation Forcing




Resonance $$\Omega_s = \frac{2}{1+(1-\eta)^2}\,|\Omega_p + \Omega_s|$$

In the limit: $\eta << 1,\ Po << 1,\ \alpha << 1$

$$Po \sim -\eta \qquad Po = \frac{\Omega_p}{\Omega_s}$$