Quasi-Geostrophic Theory From Stratified to Unstably-Stratified Flows

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Outline

- Motivation: rotationally constrained convective systems
  - Planetary + Stellar interiors
  - Turbulence problem: high Re + balanced motions
- Reduced modeling approaches
  - Asymptotically exact reductions of NSE that capture balanced geostrophic dynamics
  - Derivation for stably-stratified or unstably-stratified flows

- Incompressible fluid equations

$$\left(\partial_t + \boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} + \frac{1}{Ro} \hat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla p + \Gamma T \hat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\left(\partial_t + \boldsymbol{u} \cdot \nabla\right) T = \frac{1}{Pe} \nabla^2 T$$

- Incompressible fluid equations

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- Non-dimensional parameters (generic based on $\{L, U, L/U, P, \Delta T\}$)
- All fluid variables considered $\mathcal{O}(1)$

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$Ro = \frac{U}{2\Omega L} = \frac{\tau_\Omega}{\tau_{eddy}}$, Rossby #

$Re = \frac{UL}{\nu} = \frac{\tau_\nu}{\tau_{eddy}}$, Reynolds #

$Eu = \frac{P}{\rho_0 U^2} = \frac{\tau_{eddy}^2}{\tau_p^2}$, Euler #

$Pe = \frac{UL}{\kappa} = \frac{\tau_\kappa}{\tau_{eddy}}$, Peclet #

$\Gamma = \frac{g\alpha\Delta T L}{U^2} = \frac{H/L}{Fr^2} = \frac{\tau_{eddy}^2}{\tau_{ff}^2}$, Buoyancy #

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- Non-dimensional parameters (generic based on $\{L, U, L/U, P, \Delta T\}$)
- All fluid variables considered $\mathcal{O}(1)$
- Conservation of volume averaged energy in absence of dissipation

$$E = \frac{1}{2}\left\langle \boldsymbol{u}\cdot\boldsymbol{u} + \Gamma(T-z)^2\right\rangle_V, \quad \frac{dE}{dt} = 0$$

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- All fluid variables considered $\mathcal{O}(1)$
- Conservation of volume averaged energy in absence of dissipation

$$E = \frac{1}{2}\left\langle \boldsymbol{u} \cdot \boldsymbol{u} + \Gamma(T - z)^2 \right\rangle_V , \quad \frac{dE}{dt} = 0$$

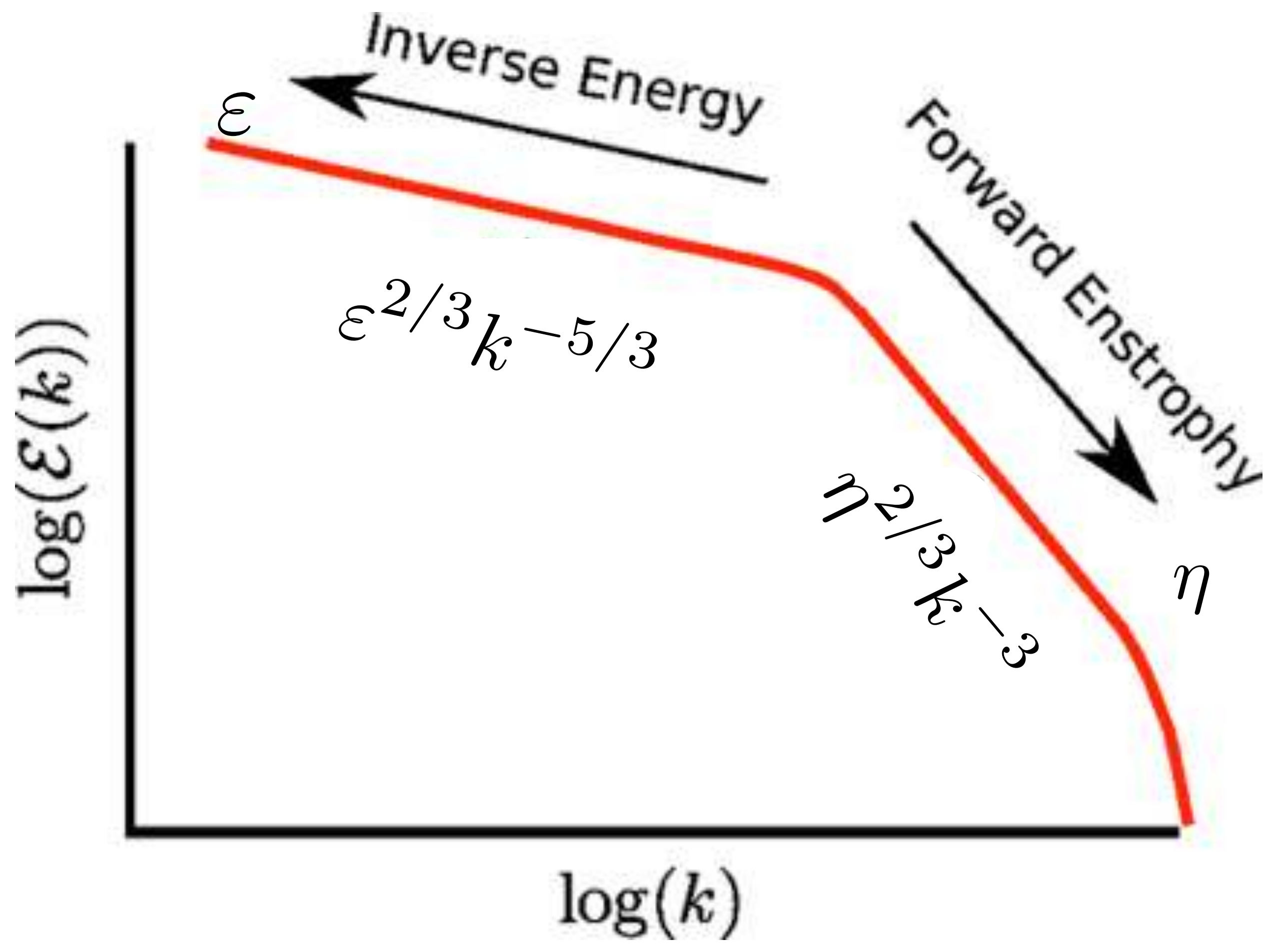
- Conservation of potential vorticity in absence of dissipation

$$q = \left(\boldsymbol{\omega} + \frac{1}{Ro}\hat{\boldsymbol{z}}\right) \cdot \nabla T, \quad \frac{Dq}{Dt} = 0$$

Nondimensional Parameters: Extreme

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{z}}\times\boldsymbol{u} = -Eu\nabla p + \Gamma T\hat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u}$$

Turbulence challenge - Dual cascade



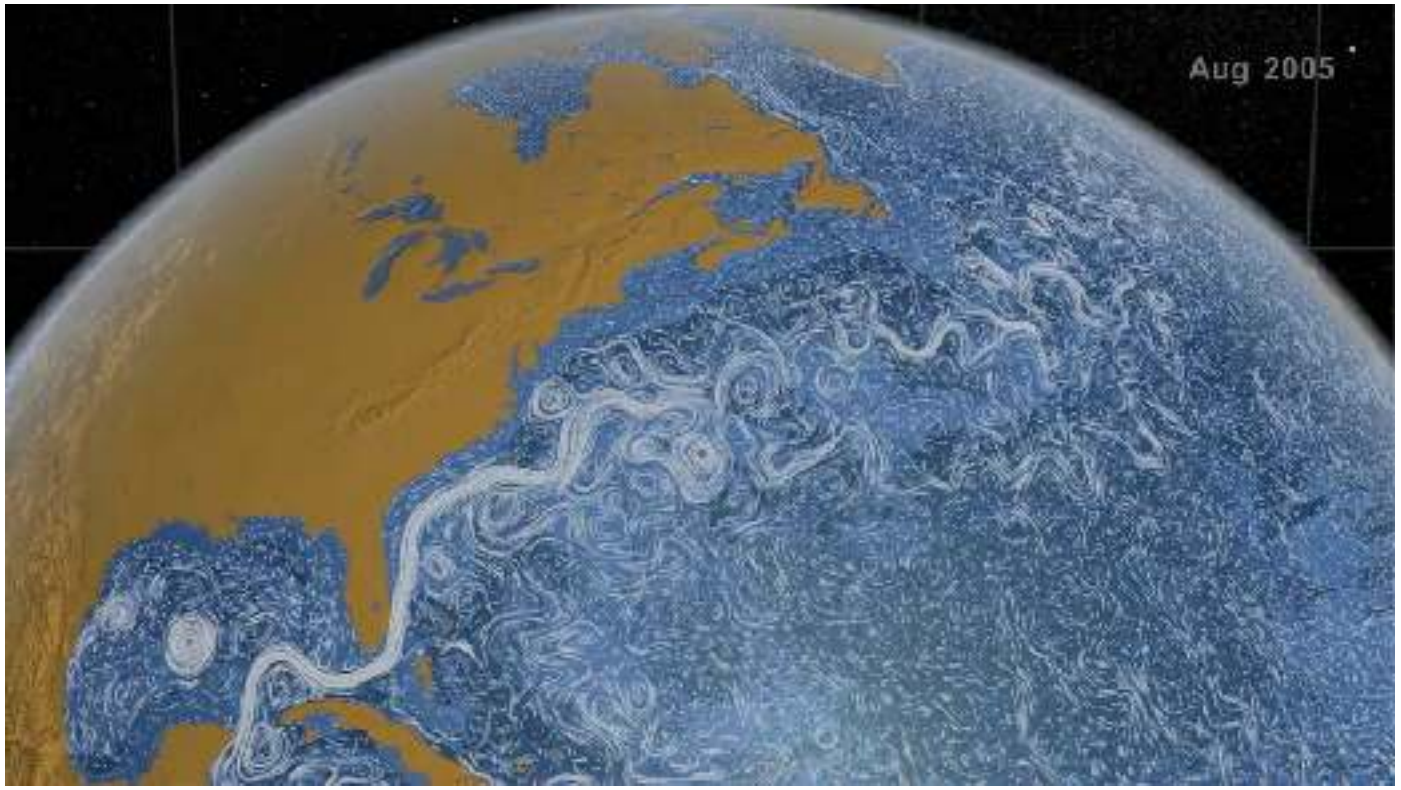
Separation of scales an issue $\quad L/l_k \sim Re^{3/4} \;\Rightarrow N^3 \sim Re^{\frac{9}{4}}$

$$(10^3)^3 \sim (10^4)^{\frac{9}{4}} \quad \Rightarrow \text{DNS}$$

$$(10^{6+})^3 \sim (10^{8+})^{\frac{9}{4}} \quad \Rightarrow \text{GAFD}$$

Rotationally Constrained Stably-Stratified Flows

$Fr = (A/\Gamma)^{1/2} \ll Ro \ll 1$



Source: NASA (GCM synthesizing satellite and in-situ data)

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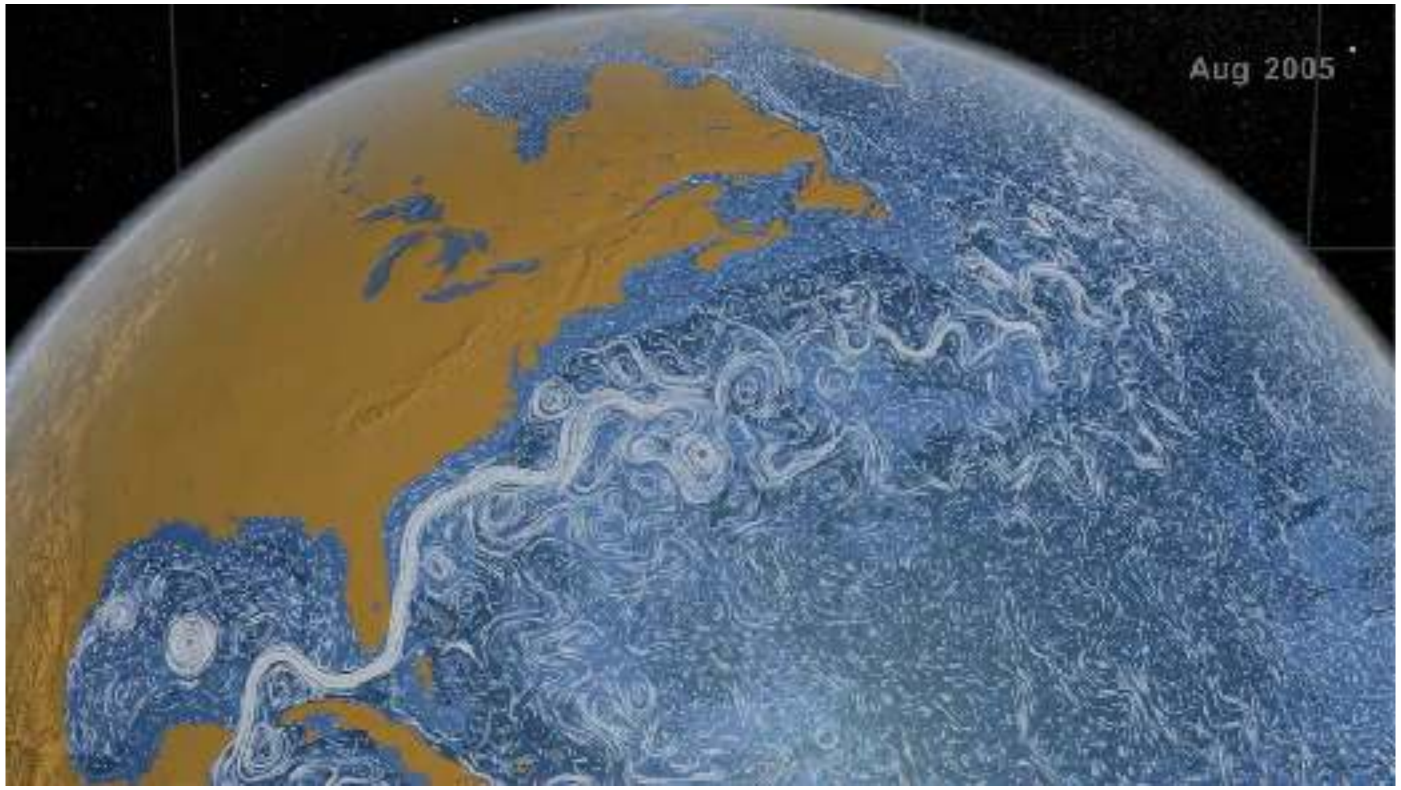
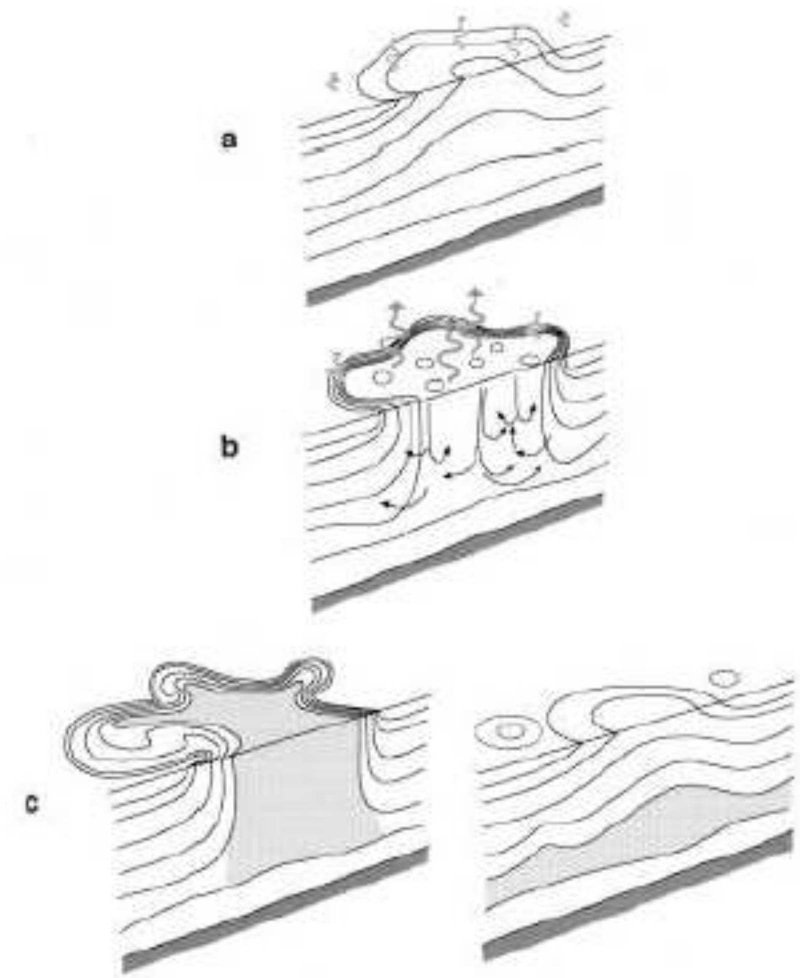
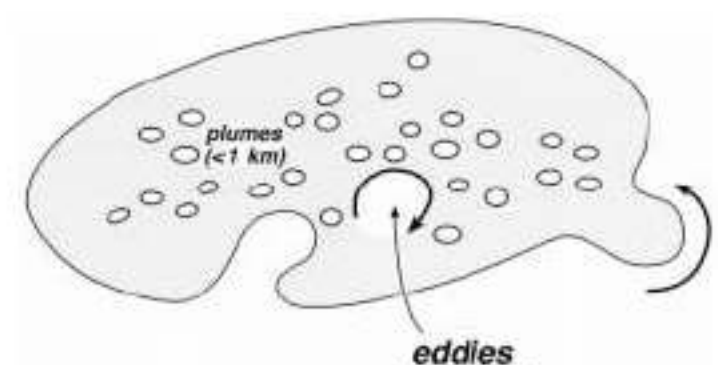


Source: NASA (GCM synthesizing satellite and in-situ data)

(Marshall and Schott, JGR 1999)





$Ro < Fr = (A/\Gamma)^{1/2} \sim 1$

Rotationally Constrained Stably-Stratified Flows

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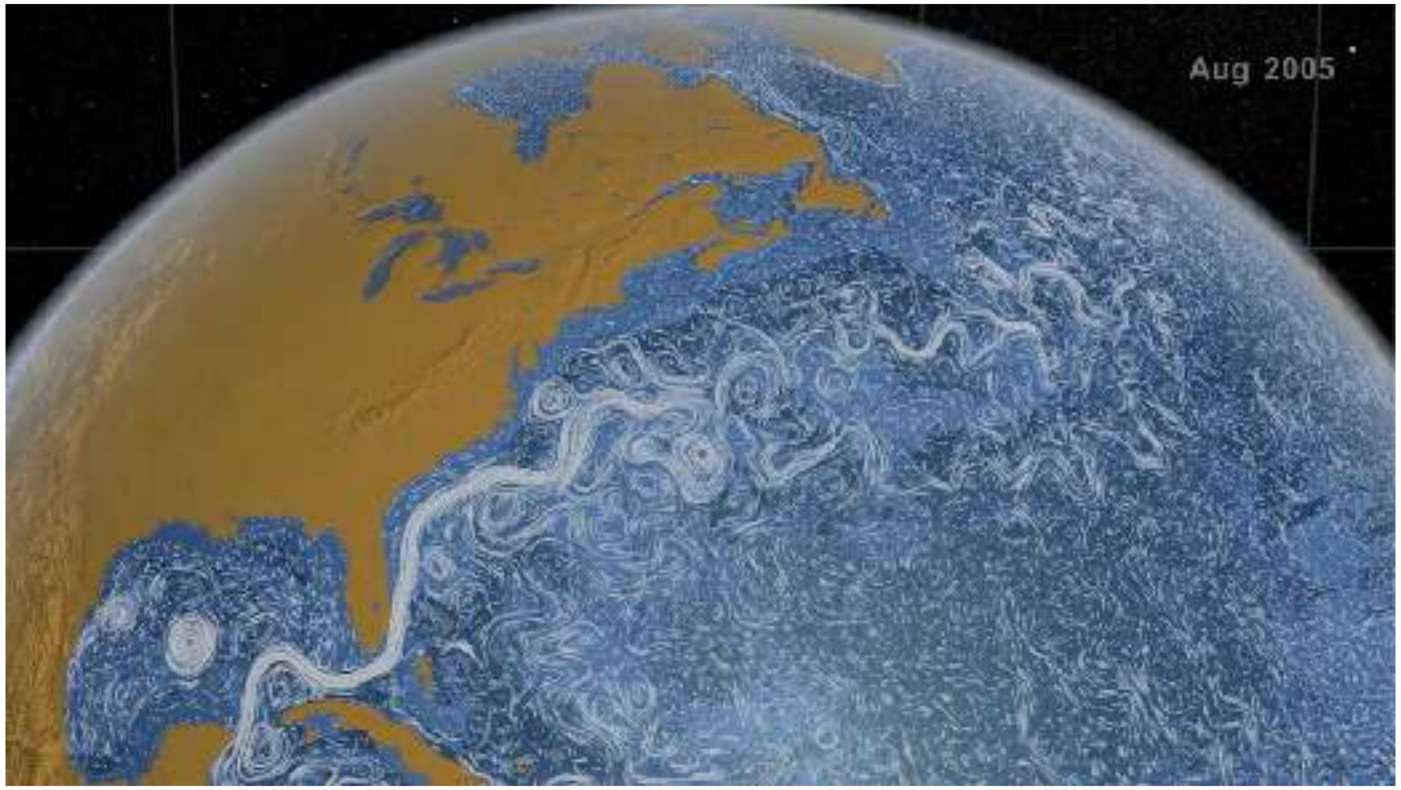
(Marshall and Schott, JGR 1999)



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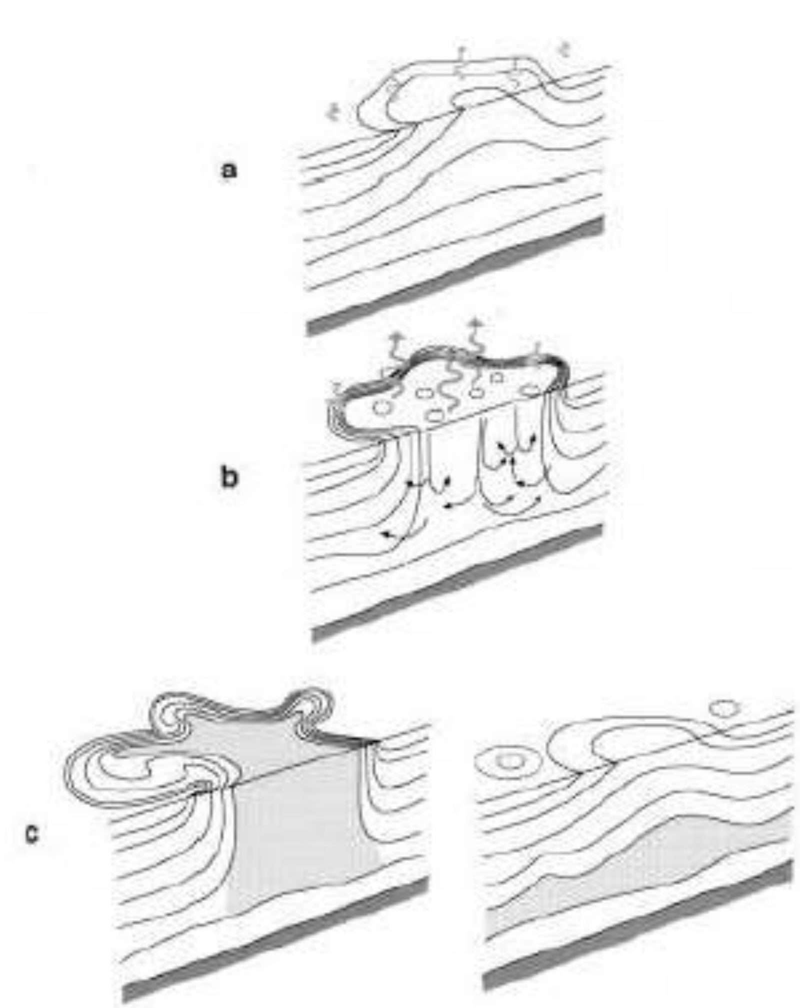
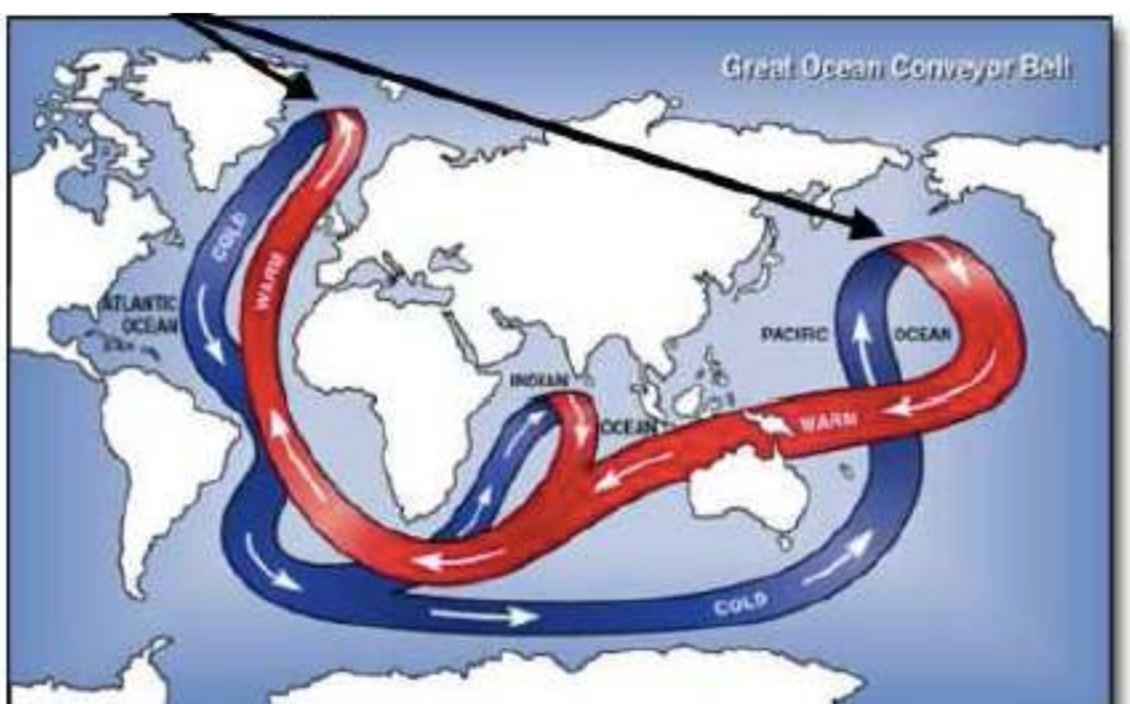
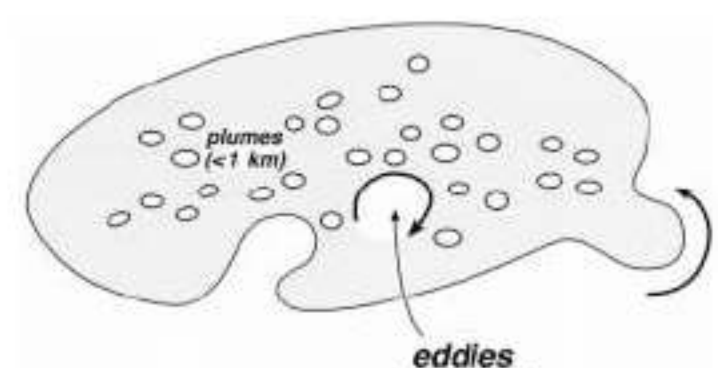


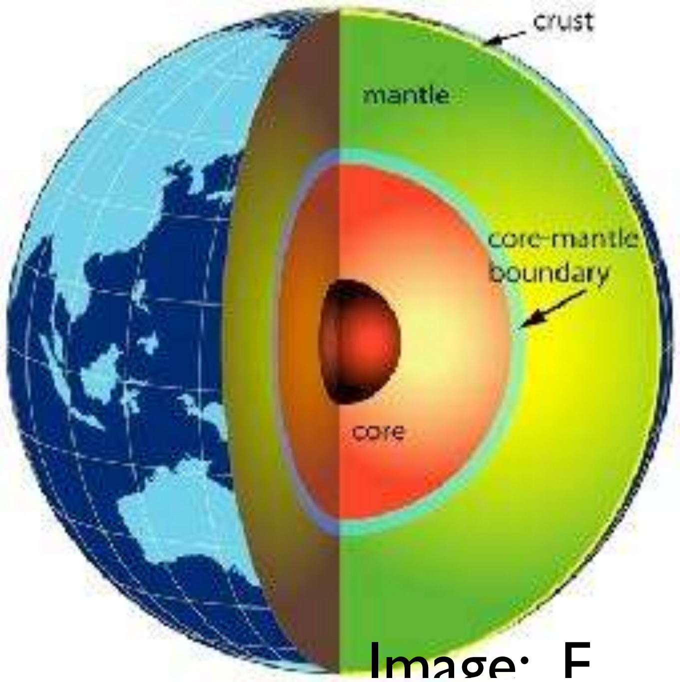




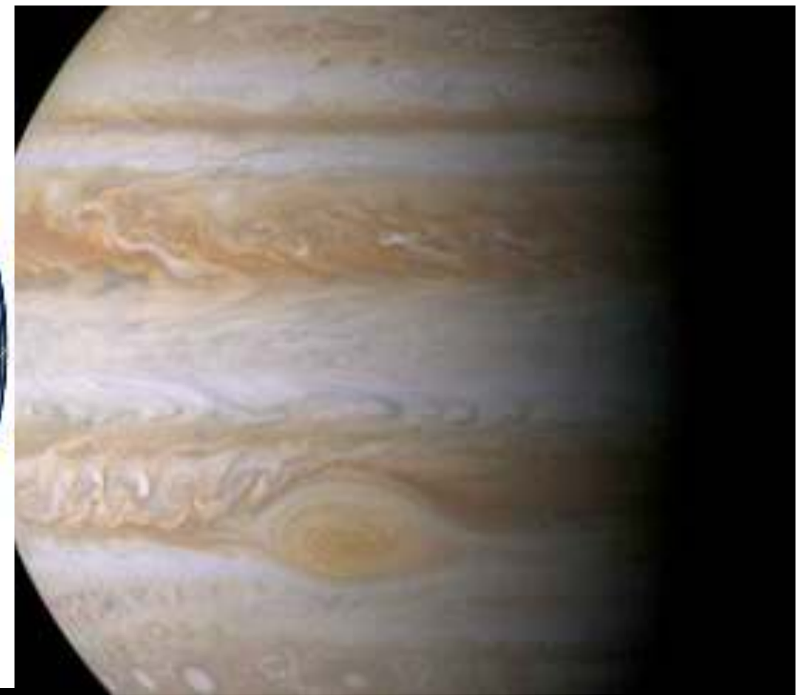
$Ro < Fr = (A/\Gamma)^{1/2} \sim 1$

Rotationally Constrained Unstably-Stratified Flows

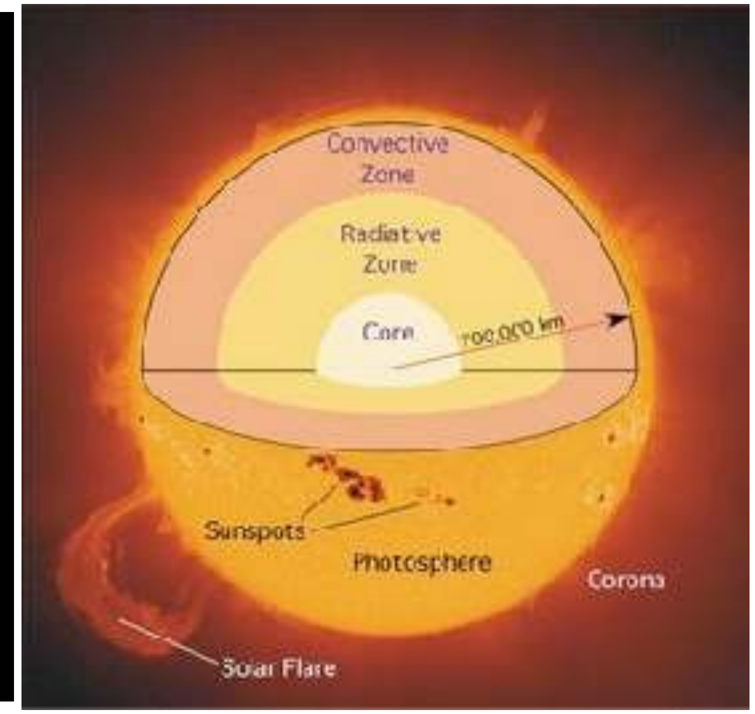
Dynamos



Zonal flows



Convective zones

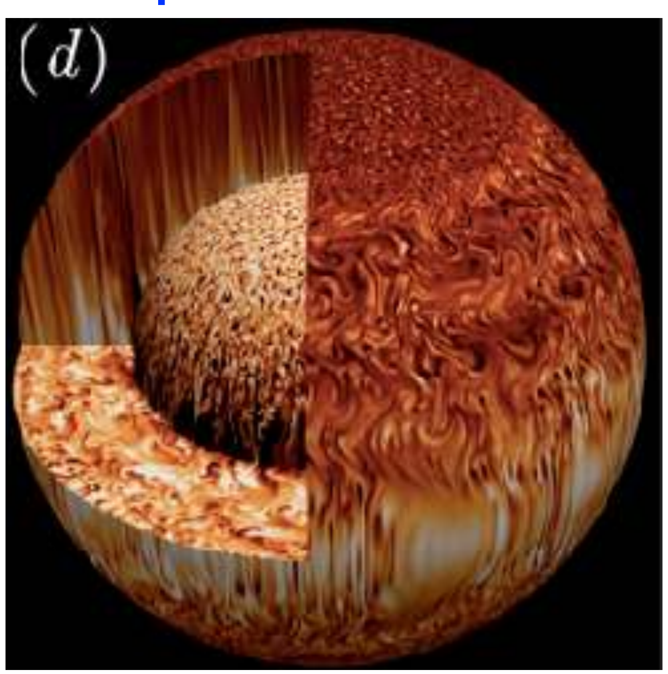


$Ro \sim 10^{-6}$

$Fr = \mathcal{O}(1)$

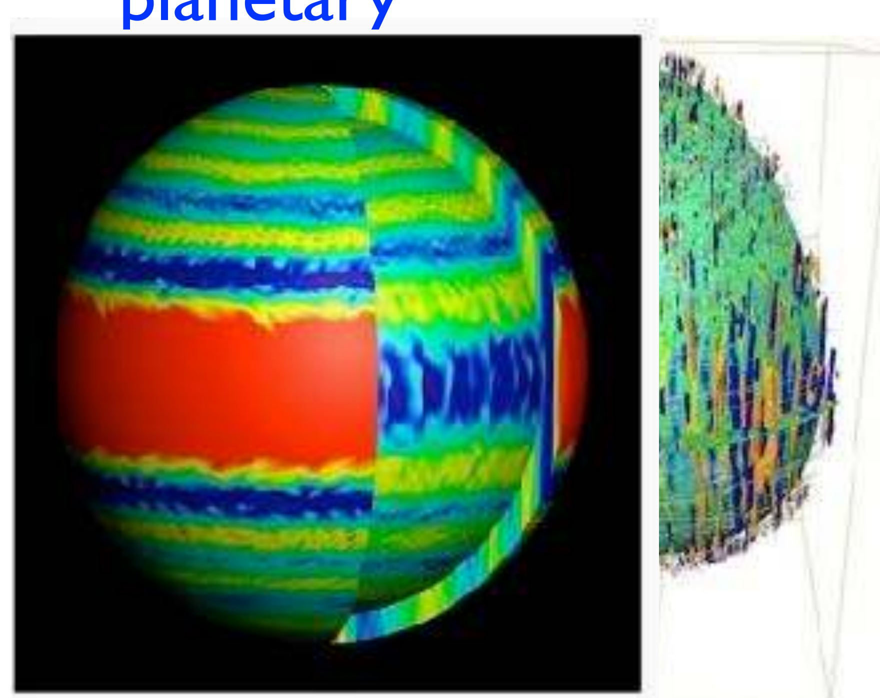
$Re \sim 10^{8^+}$

liquid outer



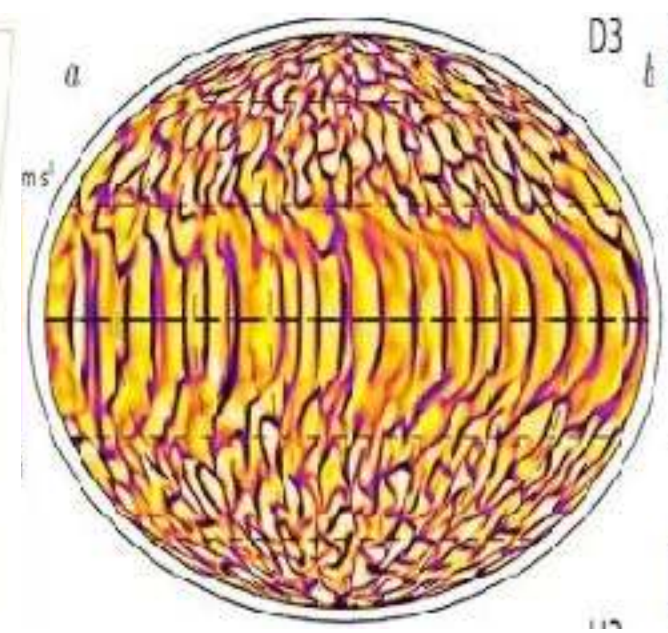
Gastine et al JFM 2016

planetary



Heimpel et al Nature 2005

Sun: Stellar



Brown et al ApJ 2008

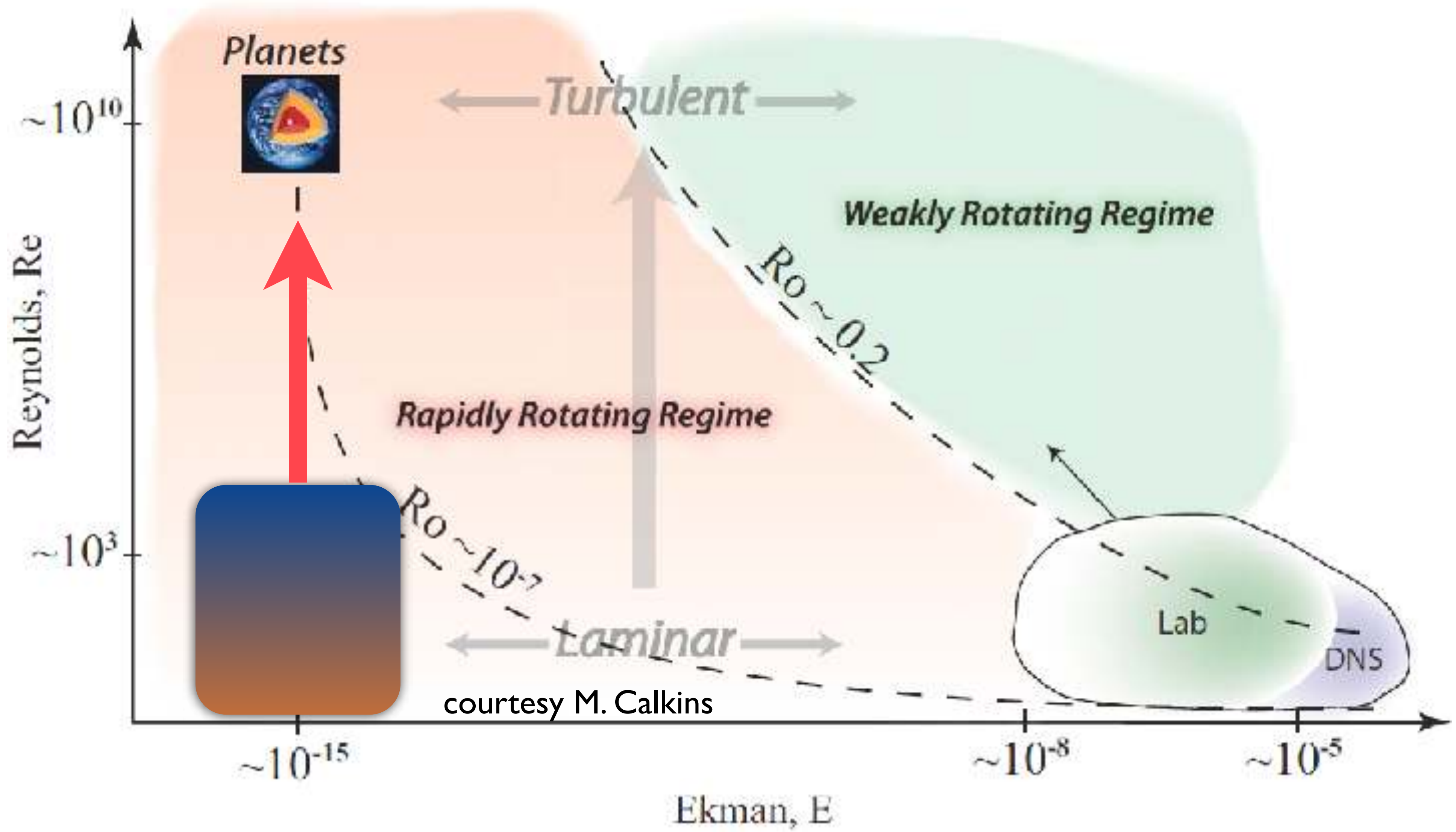
$Ro \sim 10^{-2}$

$Re \sim 10^4$

$Pr = 1$

GAFD Challenge

- Extrapolation by identifying scaling laws as a function of non-dimensional parameters
- Limitations on laboratory experiments and DNS prevent extrapolation at fixed $Ro = Re \cdot E$



$$\lim_{Re\to\infty, Ro=\epsilon} (NSE) = \text{Hard}$$

$$Re = \frac{UL}{\nu}, \quad Ro = Re \cdot E = \frac{U}{2\Omega L}$$

Reduced Quasi-Geostrophic Equations (Unified across scales)

Balance: $p = \psi_0', \quad \boldsymbol{u} = (-\partial_y \psi_0', \partial_x \psi_0', \hat{w}_0'), \quad \zeta_0' = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_0' = \nabla_\perp^2 \psi_0', \qquad T = \overline{T}(Z) + Ro\, \hat{\vartheta}_0'.$

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Fluctuating Equations:

Vert. Vorticity

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\ \zeta_0' - \partial_Z \hat{w}_0' = \frac{1}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \zeta_0'$$

Vert. Velocity

$$(ARo)^2 (\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{w}_0' + \partial_Z \psi_0' = \widetilde{\Gamma} \hat{\vartheta}_0' + \frac{(ARo)^2}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{w}_0'$$

Temp. Fluct.

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{\vartheta}_0' + \hat{w}_0' \partial_Z \overline{T}_0 = \frac{1}{Pe} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{\vartheta}_0'$$

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Mean Equations:

Hydrostatic balance

$$\partial_Z \overline{p}_0 = \widetilde{\Gamma}\overline{T}_0$$

Mean temp

$$\frac{1}{Ro^2}\partial_t \overline{T} + \partial_Z(\overline{\hat{w}_0' \hat{\vartheta}_0'}) = \frac{1}{(ARo)^2 Pe}\partial_{ZZ}\overline{T}_0$$

Reduced Quasi-Geostrophic Equations (Unified across scales)

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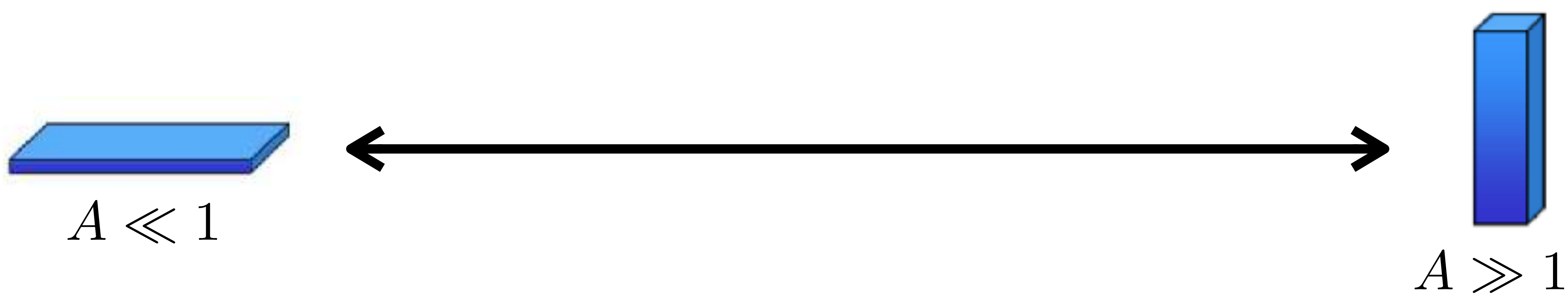

Reduced Quasi-Geostrophic Equations (strong stratification)

Balance: $p = \psi_0', \quad \boldsymbol{u} = (-\partial_y \psi, \partial_x \psi, \hat{w}_0'), \quad \zeta_0' = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_0' = \nabla_\perp^2 \psi_0', \quad T = \overline{T}(Z) + Ro\, \hat{\vartheta}_0'.$

Fluctuating Equations:

Vert. Vorticity $(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\, \zeta_0' - \partial_Z \hat{w}_0' = 0$

Vert. Velocity $\partial_Z \psi_0' = \widetilde{\Gamma} \hat{\vartheta}_0'$

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) q_0' = 0$$

$$q_0' = \nabla_\perp^2 \psi_0' - \partial_Z \left(\frac{1}{\widetilde{\Gamma} \partial_Z \overline{T}} \partial_Z \psi_0' \right)$$

Temp. Fluct. $(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{\vartheta}_0' + \hat{w}_0' \partial_Z \overline{T}_0 = 0$

$A \ll 1$

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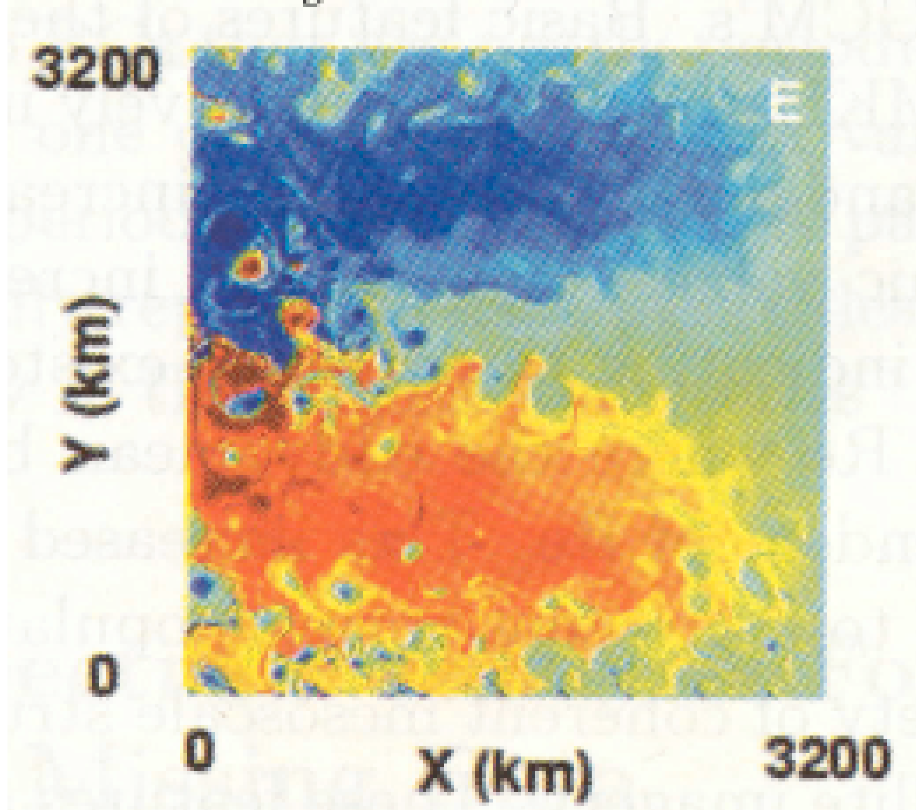
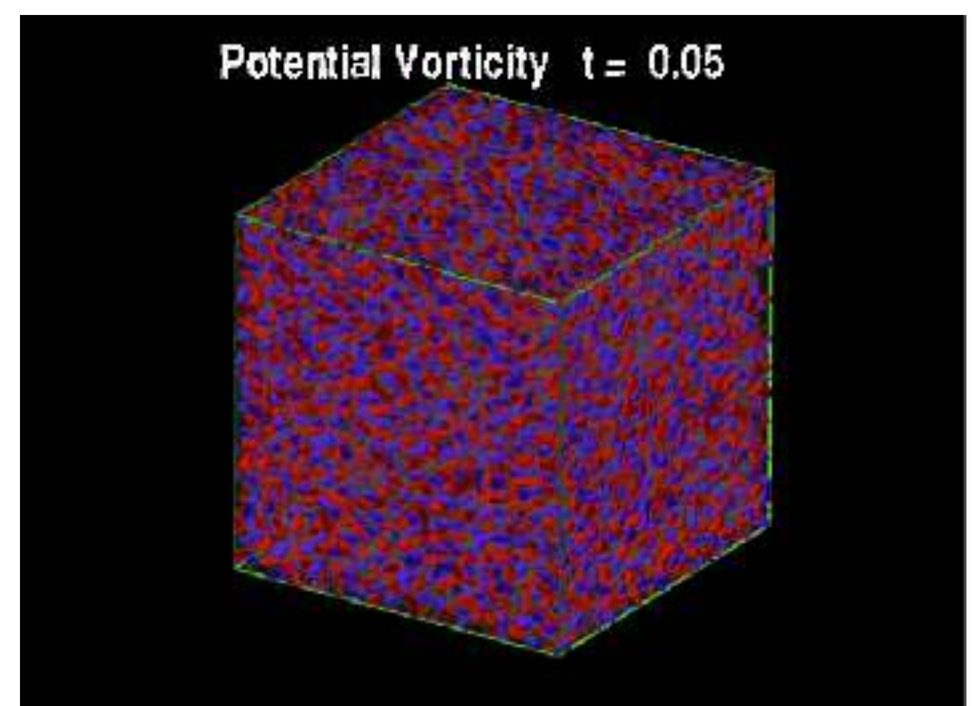
Siegel et al GRL 2001



Courtesy Jeff Weiss



Reduced Quasi-Geostrophic Equations (Unified)

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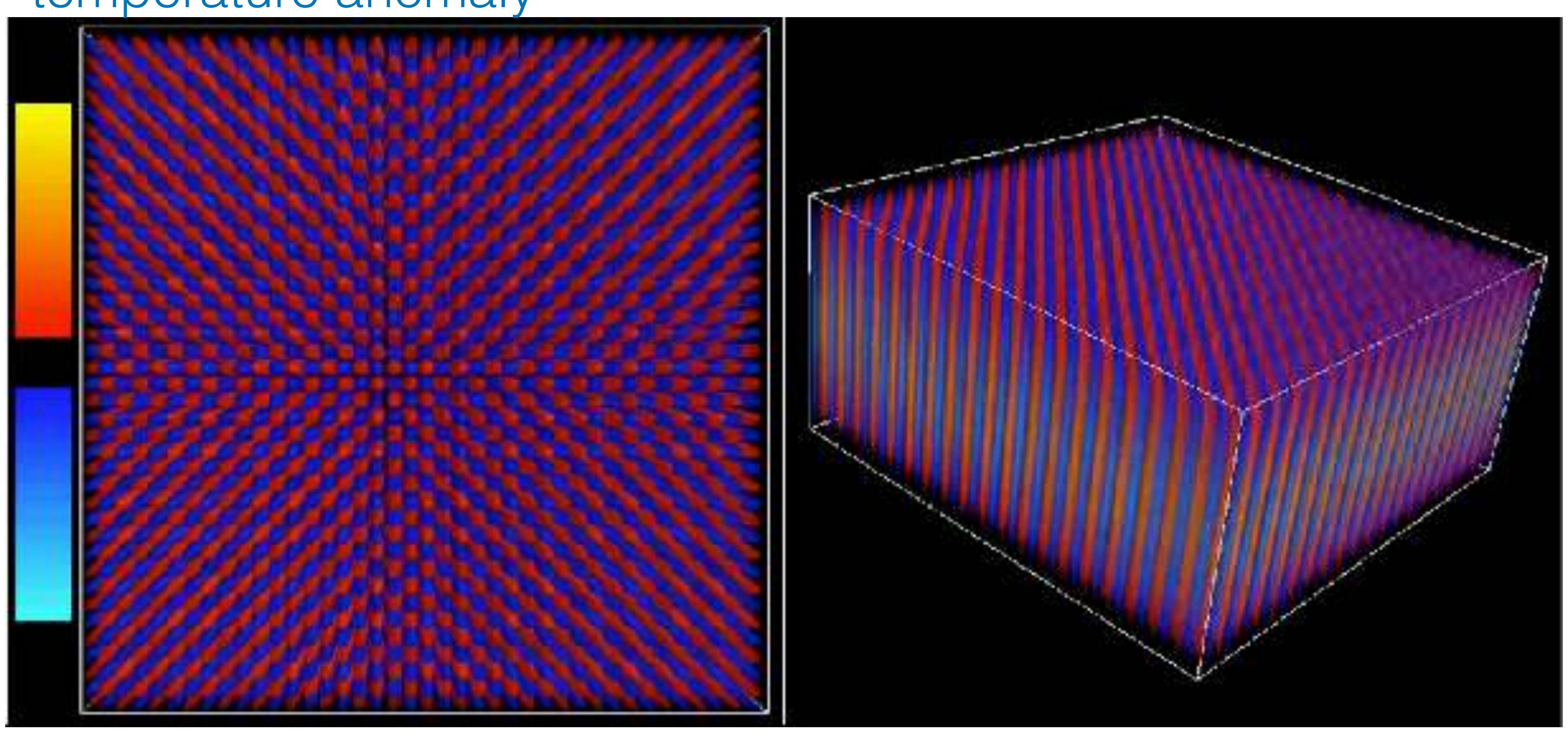
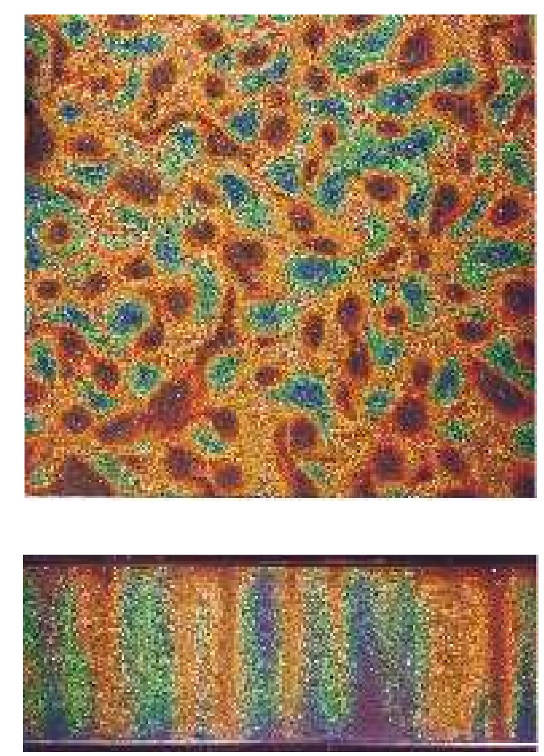
temperature anomaly

$A \gg 1$



$RaE^{4/3} = 40, \sigma = 7$

Sprague et al JFM '06



Sakai JFM 1997

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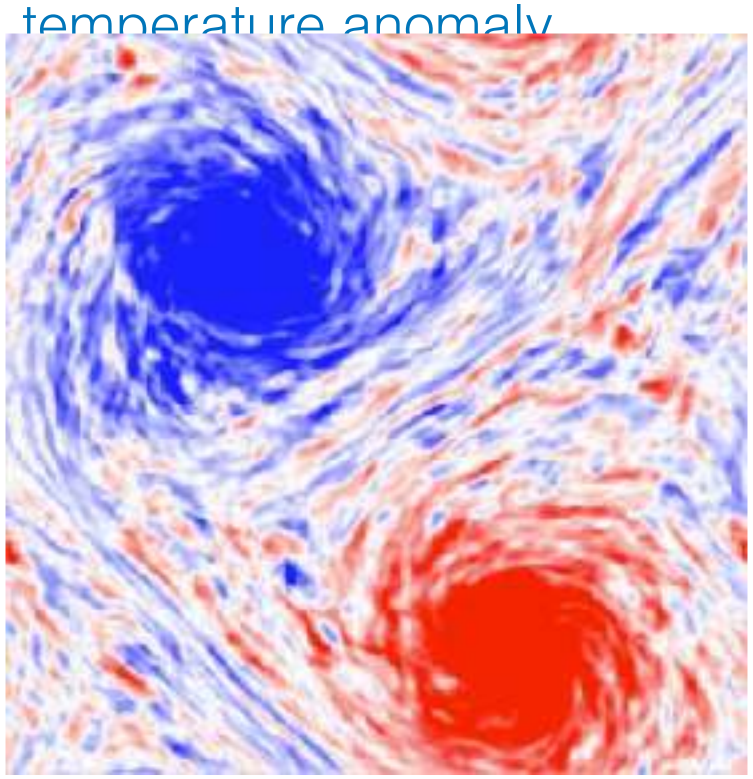
Temp. Fluct.

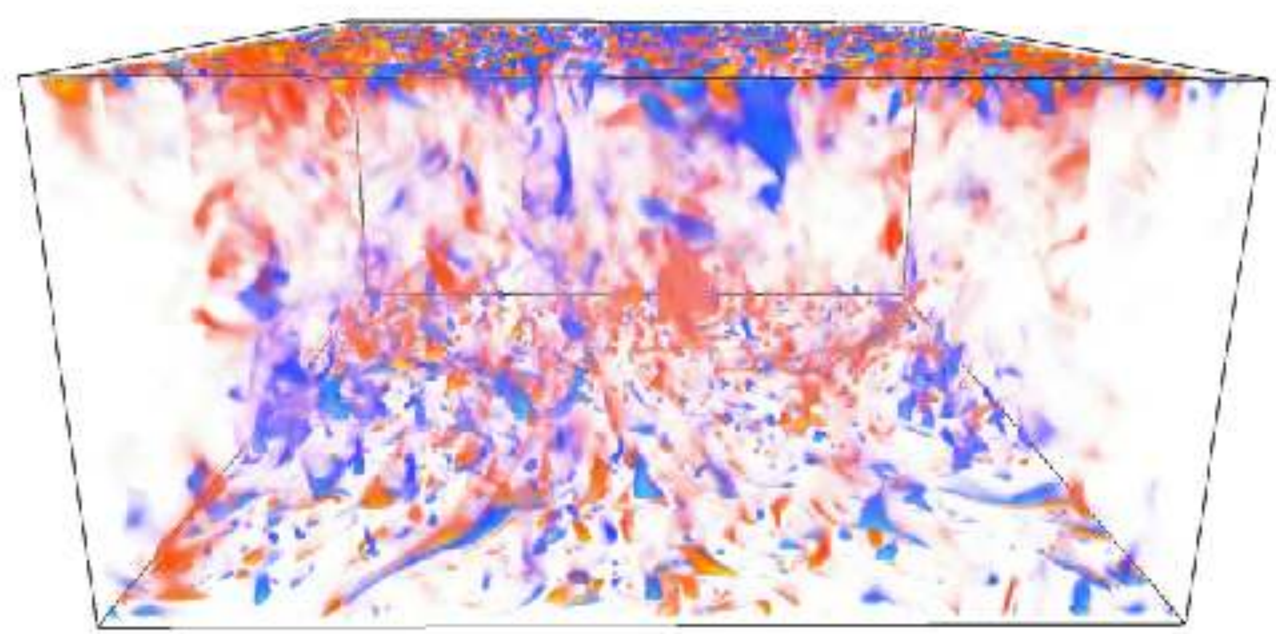
$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{\vartheta}_0' + \hat{w}_0' \partial_Z \overline{T}_0 = \frac{1}{Re} \left(\nabla_\perp \cdot + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{\vartheta}_0'$$

$A \gg 1$

temperature anomaly



$RaE^{4/3} = 160 \;\; \sigma = 1$



Rubio al PRL '14

- Decompose into horizontally averaged (mean) & fluctuating components

$$f(\boldsymbol{x}_\perp, z, t) = \overline{f}(z,t) + f'(\boldsymbol{x}_\perp, z, t) \qquad \text{where} \qquad \overline{f} = \frac{1}{A}\int f dxdy, \quad \overline{f'} = 0$$

$$T = \overline{T} + \vartheta'$$

- Averaged continuity equations implies zero vertical mass flux

$$\overline{\nabla \cdot \boldsymbol{u}} = \partial_z \overline{w} = 0 \quad \Rightarrow \quad \overline{w} = 0 \qquad \overline{\boldsymbol{u}} = \overline{\boldsymbol{u}}_\perp = (\overline{u}, \overline{v}, 0)$$

Decomposed NSE

- Mean Equations:

$$\partial_t \overline{\boldsymbol{u}}_\perp + \partial_z \left(\overline{w' \boldsymbol{u'}} \right) + \frac{1}{Ro} \hat{\boldsymbol{z}} \times \overline{\boldsymbol{u}}_\perp = -Eu \partial_z \overline{p} \hat{\boldsymbol{z}} + \Gamma \overline{T} \hat{\boldsymbol{z}} + \frac{1}{Re} \partial_{zz} \overline{\boldsymbol{u}}_\perp$$

$$\partial_t \overline{T} + \partial_z \left(\overline{w' \vartheta'} \right) = \frac{1}{Pe} \partial_{zz} \overline{T}$$

- Fluctuating Equations:

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Rotationally Constrained Flows - Geostrophic Balance $Ro \ll 1$

Geostrophy as the leading order balance

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Aurnou UCLA SpinLab

Fluid motions are inherently columnar

Strongly constrained GAFD flows are not 2D. ? How is 3D recaptured while preserving geostrophy

Rotationally constrained flows and aspect ratio

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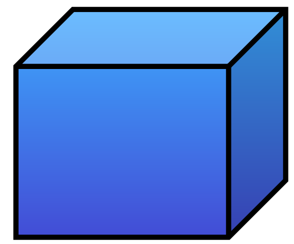
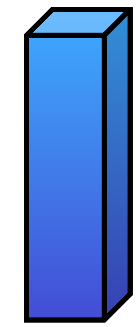
- Reduction of NSE for geostrophically balanced flows exists across all scales.

	QG	**Intermediate**	**Convection**
			
$A =$	$H/L << 1$	$H/L = O(1)$	$H/L >> 1$
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	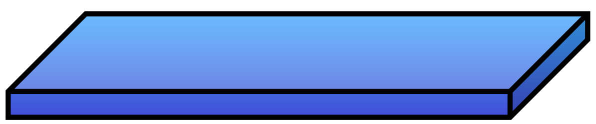	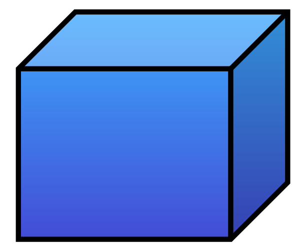	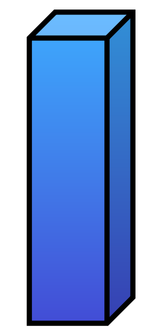
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		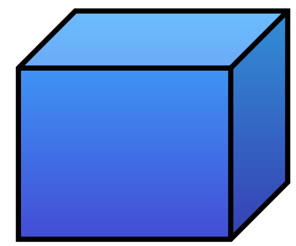	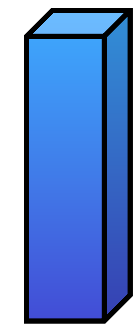
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	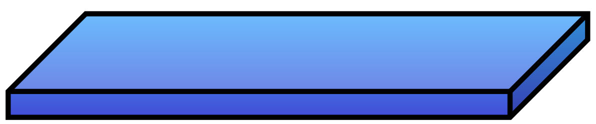	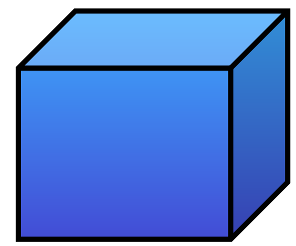	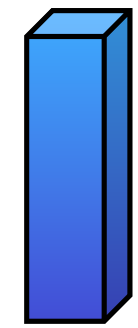
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	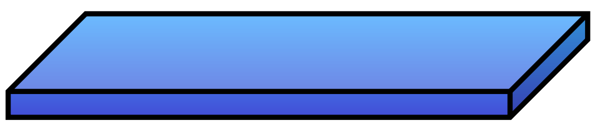	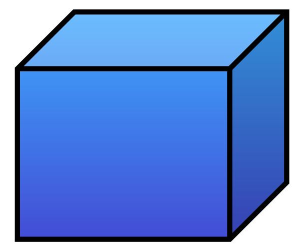	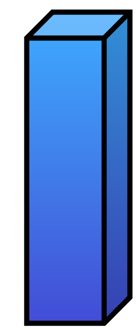
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Anistropically-scaled NSE

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- Utilize observations (often data-poor), laboratory exp's & simulations (often not in GAFD limit), theory (when it exists) or straight trial by fire

$$Ro = \varepsilon \ll 1$$

Necessary Conditions from Fluctuating Equations

$$\left(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp\right) \boldsymbol{u}'_\perp + \frac{1}{A} w' \partial_Z \boldsymbol{u}_\perp - \frac{1}{A}\partial_Z \left(\overline{w' \boldsymbol{u}'_\perp}\right) + \frac{1}{Ro} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp = -Eu \nabla_\perp p' + \frac{1}{Re} \left(\nabla^2_\perp + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z\right)^2 \boldsymbol{u}'_\perp$$

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$$\frac{1}{Ro}\hat{\boldsymbol{z}} \cdot \nabla \times \left(\hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp\right) \sim (\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,(\hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}) \sim \mathcal{O}(1)$$

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- Universal velocity scaling $w' \sim ARo$ s.t. $w' = ARo\,\hat{w}'$

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	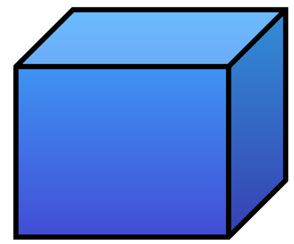	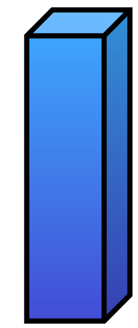
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$$ARo^2\partial_Z \left(\overline{\hat{w}'\hat{w}'}\right) = -\frac{Eu}{A}\partial_Z\overline{p} + \Gamma\overline{T}$$

$$\partial_t \overline{T} + Ro\partial_Z \left(\overline{\hat{w}'\vartheta'}\right) = \frac{1}{A^2 Pe}\partial_{ZZ}\overline{T}$$

- Mean hydrostatic support $Eu \sim \Gamma A$, hydrostatic balance if $ARo^2 < \Gamma$
- Mean flow driven by Reynolds stresses: $\overline{\boldsymbol{u}}_\perp \ll \boldsymbol{u}'_\perp = \mathcal{O}(1)$
- Dissipation subdominant if $Re > \dfrac{Ro}{A^2}$

$$Ro = \varepsilon \ll 1$$

Updated Fluctuating Equations

- Given planetary vortex stretching constraint $w' = ARo\,\hat{w}'$

- Fluctuating Equations

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\boldsymbol{u}'_\perp + Ro\,\hat{w}'\partial_Z \boldsymbol{u}_\perp - Ro\,\partial_Z\left(\overline{\hat{w}'\boldsymbol{u}'_\perp}\right) + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp = -Eu\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_\perp$$

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\hat{w}' + Ro\,\hat{w}'\partial_Z \hat{w}' - Ro\,\partial_Z\left(\overline{\hat{w}'\hat{w}'}\right) = -\frac{Eu}{A^2 Ro}\partial_Z p' + \frac{\Gamma}{ARo}\vartheta' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'$$

$$\nabla_\perp \cdot \boldsymbol{u}'_\perp + Ro\,\partial_Z \hat{w}' = 0$$

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\vartheta' + Ro\hat{w}'\partial_Z T - Ro\partial_Z\left(\overline{\hat{w}'\vartheta'}\right) = \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \vartheta'$$

- Observations:
  - Vertical advection of momenta subdominant in prognostic dynamics (hallmark of QG theory)
  - Flows are largely horizontally non-divergent.

Updated Fluctuating Equations

$Ro = \varepsilon \ll 1$

- Given planetary vortex stretching constraint $w' = ARo\,\hat{w}'$

- Fluctuating Equations

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\boldsymbol{u}'_\perp + \cancel{Ro\,\hat{w}'\partial_Z \boldsymbol{u}_\perp} - \cancel{Ro\,\partial_Z\left(\overline{\hat{w}'\boldsymbol{u}'_\perp}\right)} + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp = -Eu\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_\perp$$

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\hat{w}' + \cancel{Ro\,\hat{w}'\partial_Z \hat{w}'} - \cancel{Ro\,\partial_Z\left(\overline{\hat{w}'\hat{w}'}\right)} = -\frac{Eu}{A^2 Ro}\partial_Z p' + \frac{\Gamma}{ARo}\vartheta' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'$$

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$$Ro = \varepsilon \ll 1$$

Updated Fluctuating Equations

- Given planetary vortex stretching constraint $w' = ARo\,\hat{w}'$

- Fluctuating Equations

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\boldsymbol{u}'_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -Eu\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_\perp$$

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- Observations:
  - Vertical advection of momenta subdominant in prognostic dynamics (hallmark of QG theory)
  - Flows are largely horizontally non-divergent.

$$Ro = \varepsilon \ll 1$$

Additional Conditions from Fluctuating Equations

- Given $\overline{\boldsymbol{u}}_\perp \ll \boldsymbol{u}'_\perp = \mathcal{O}(1)$

- Fluctuating Equations

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\boldsymbol{u}'_\perp + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -Eu\nabla_\perp p' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_\perp$$

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\,\hat{w}' \approx -\frac{Eu}{A^2 Ro}\partial_Z p' + \frac{\Gamma}{ARo}\vartheta' + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'$$

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- Geostrophic balance: $p' \sim \dfrac{1}{EuRo}$

- Potential energy reservoir: $\vartheta' \sim Ro$

- Hydrostatic support: $\Gamma\vartheta' \sim \dfrac{1}{ARo} \quad \Rightarrow \quad \boxed{\Gamma \sim \dfrac{1}{ARo^2}}$

$$Ro = \varepsilon \ll 1$$

Additional Conditions from Fluctuating Equations

- Given $\overline{\boldsymbol{u}}_\perp \ll \boldsymbol{u}'_\perp = \mathcal{O}(1)$

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$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\, \boldsymbol{u}'_\perp + \boxed{\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp \approx -Eu\nabla_\perp p'} + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_\perp$$

$$(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\, \hat{w}' \approx \boxed{-\frac{Eu}{A^2 Ro}\partial_Z p' + \frac{\Gamma}{ARo}\vartheta'} + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'$$

$$\nabla_\perp \cdot \boldsymbol{u}'_\perp + Ro\, \partial_Z \hat{w}' = 0$$

$$\boxed{(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\, \vartheta' + Ro \hat{w}' \partial_Z \overline{T}} \approx \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \vartheta'$$

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Updated Mean Equations

$Ro = \varepsilon \ll 1$

- Given $\Gamma \sim \frac{1}{ARo^2}$ resulting from mean hydrostatic support

- Mean Equations:

$$\partial_t \overline{\boldsymbol{u}}_\perp + Ro\partial_Z \left(\overline{\hat{w}'\boldsymbol{u}'_\perp}\right) + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \overline{\boldsymbol{u}}_\perp = \frac{1}{A^2 Re}\partial_{ZZ}\overline{\boldsymbol{u}}_\perp$$

$$ARo^2\partial_Z \left(\overline{\hat{w}'\hat{w}'}\right) = -\frac{Eu}{A}\partial_Z \overline{p} + \Gamma\overline{T}$$

$$\partial_t \overline{T} + Ro\partial_Z \left(\overline{\hat{w}'\vartheta'}\right) = \frac{1}{A^2 Pe}\partial_{ZZ}\overline{T}$$

- Mean hydrostatic support $Eu \sim \Gamma A$, $\Rightarrow$ $Eu \sim \frac{1}{Ro^2}$

- Mean thermal dissipation bounded $Pe, Re \gtrsim \frac{1}{A^2 Ro^2} \gtrsim 1 \quad \Rightarrow \quad A \lesssim \frac{1}{Ro}$

- Hydrostatic balance only $ARo^2 \ll \Gamma$

Updated Mean Equations

$$Ro = \varepsilon \ll 1$$

- Given $\Gamma \sim \frac{1}{ARo^2}$ resulting from mean hydrostatic support

- Mean Equations:

$$\partial_t \overline{\boldsymbol{u}}_\perp + Ro\partial_Z \left(\overline{\hat{w}'\boldsymbol{u}'_\perp}\right) + \frac{1}{Ro}\hat{\boldsymbol{z}} \times \overline{\boldsymbol{u}}_\perp = 0$$

$$0 = -\frac{Eu}{A}\partial_Z \overline{p} + \Gamma \overline{T}$$

$$\partial_t \overline{T} + Ro\partial_Z \left(\overline{\hat{w}'\vartheta'}\right) = \frac{1}{A^2 Pe}\partial_{ZZ}\overline{T}$$

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- Hydrostatic balance only $ARo^2 \ll \Gamma$

Summary

Distinguished Limit

Geostrophy

Planetary vortex stretching

Hydrostatic Support

Mean thermal dissip'n bound

$$Eu = \frac{1}{Ro^2}$$

$$A \lesssim \frac{1}{Ro}$$

$$\Gamma = \frac{1}{ARo^2}\widetilde{\Gamma}, \quad \widetilde{\Gamma} = \mathcal{O}(1)$$

$$Pe, Re \gtrsim \frac{1}{A^2 Ro^2}$$

Fluid Variables

Mean:

$$\overline{p}, \overline{T} = \mathcal{O}(1)$$

$$\overline{\boldsymbol{u}}_\perp = \mathcal{O}(Ro^2), \quad \overline{w} = 0$$

Fluctuating:

$$p', \vartheta' = \mathcal{O}(Ro)$$

$$\boldsymbol{u}'_\perp = \mathcal{O}(1)$$

$$w' = ARo$$

Summary

Distinguished Limit

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$$Eu = \frac{1}{Ro^2}$$

$$A \lesssim \frac{1}{Ro}$$

$$\Gamma = \frac{1}{ARo^2}\widetilde{\Gamma}, \quad \widetilde{\Gamma} = \mathcal{O}(1)$$

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Fluid Variables

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$$\overline{p}, \overline{T} = \mathcal{O}(1)$$

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Fluctuating:

$$p', \vartheta' = \mathcal{O}(Ro)$$

$$\boldsymbol{u}'_\perp = \mathcal{O}(1)$$

$$w' = ARo$$

Asymptotic Perturbation Theory: Fluctuating Equations

$$\mathcal{L}_{geo}\boldsymbol{v'} = \boldsymbol{R}$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u'}_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u'}_\perp \\ 0 \end{pmatrix} \qquad \boldsymbol{R} = Ro \begin{pmatrix} -\left((\partial_t + \boldsymbol{u'}_\perp \cdot \nabla_\perp)\, \boldsymbol{u'}_\perp - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u'}_\perp \right) \\ -\partial_Z \hat{p}' + \hat{\vartheta}' - A^2 Ro \left((\partial_t + \boldsymbol{u'}_\perp \cdot \nabla_\perp)\, \hat{w}' - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}' \right) \\ -\partial_Z \hat{w}' \\ -(\partial_t + \boldsymbol{u}_\perp \cdot \nabla_\perp)\, \hat{\vartheta}' - \hat{w}'\partial_Z \overline{T} + \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}' \end{pmatrix}$$

- Introduce asymptotic expansions $\quad \hat{\boldsymbol{v}}' = (\hat{\boldsymbol{u}}'_\perp, \hat{w}', \hat{p}', \hat{\vartheta}') = \hat{\boldsymbol{v}}'_{\boldsymbol{0}} + Ro\, \hat{\boldsymbol{v}}'_{\boldsymbol{1}} + \cdots$
- Solve sequence of linear homogeneous PDE's:

$$\mathcal{O}(1): \quad \mathcal{L}_{geo}\boldsymbol{v}'_0 = \boldsymbol{0}$$

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}'_1 = \boldsymbol{R}_1$$

$$\mathcal{O}(Ro^2): \quad \mathcal{L}_{geo}\boldsymbol{v}'_2 = \boldsymbol{R}_2$$

- Ensure solution remain bounded in space and time at all orders

Asymptotic Perturbation Theory: Leading order

$$\mathcal{O}(1): \qquad \mathcal{L}_{geo}\boldsymbol{v}_0' = \mathbf{0}$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}'_\perp \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_0' = (u_0', v_0', \hat{w}_0', \hat{p}_0', \hat{\vartheta}_0') = (-\partial_y \psi_0', \partial_x \psi_0', \hat{w}_0', \psi_0', \hat{\vartheta}_0')$$

- Diagnostic solution w linear relationship btw pressure & horizontal velocity field
- Comment: If vertical modulation was permissible as in columnar regimes Proudman-Taylor constraint would eliminate them.

Asymptotic Perturbation Theory: First Order

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}_1' = \boldsymbol{R}_1$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}'_\perp \\ 0 \end{pmatrix} \qquad \boldsymbol{R}_1 = \begin{pmatrix} -\left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \boldsymbol{u}'_{0\perp} - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_{0\perp} \right) \\ -\partial_Z \hat{p}'_0 + \hat{\vartheta}'_0 - A^2 Ro^2 \left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \hat{w}'_0 - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'_0 \right) \\ -\partial_Z \hat{w}'_0 \\ -(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \hat{\vartheta}'_0 - \hat{w}'_0 \partial_Z \overline{T}_0 + \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}'_0 \end{pmatrix}$$

- Requirement $\boldsymbol{v}_1'$ bounded
- IFF all resonant terms in $\boldsymbol{R}_1$ are projected out

Asymptotic Perturbation Theory: First Order

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}_1' = \boldsymbol{R}_1$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}'_\perp \\ 0 \end{pmatrix} \qquad \boldsymbol{R}_1 = \begin{pmatrix} -\left(\left(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp\right) \boldsymbol{u}'_{0\perp} - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_{0\perp} \right) \\ -\partial_Z \hat{p}'_0 + \hat{\vartheta}'_0 - A^2 Ro^2 \left(\left(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp\right) \hat{w}'_0 - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'_0 \right) \\ -\partial_Z \hat{w}'_0 \\ -\left(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp\right) \hat{\vartheta}'_0 - \hat{w}'_0 \partial_Z \overline{T}_0 + \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}'_0 \end{pmatrix}$$

- Find $\boldsymbol{v}^\dagger$ orthogonal to the span of $\mathcal{L}_{geo}\boldsymbol{v}'$. Then

Asymptotic Perturbation Theory: First Order

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}_1' = \boldsymbol{R}_1$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}'_\perp \\ 0 \end{pmatrix} \qquad \boldsymbol{R}_1 = \begin{pmatrix} -\left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \boldsymbol{u}'_{0\perp} - \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}'_{0\perp} \right) \\ -\partial_Z \hat{p}'_0 + \hat{\vartheta}'_0 - A^2 Ro \left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \hat{w}'_0 - \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}'_0 \right) \\ -\partial_Z \hat{w}'_0 \\ -\left(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp\right) \hat{\vartheta}'_0 - \hat{w}'_0 \partial_Z \overline{T}_0 + \frac{1}{Pe} \left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}'_0 \end{pmatrix}$$

- Find $\boldsymbol{v}^\dagger$ orthogonal to the span of $\mathcal{L}_{geo}\boldsymbol{v}'$. Then

$$\left\langle \boldsymbol{v}^\dagger, \mathcal{L}_{geo}\boldsymbol{v}_1' \right\rangle_V = 0 \quad \Rightarrow \quad \left\langle \boldsymbol{v}^\dagger, \boldsymbol{R}_1 \right\rangle_V = 0$$

solvability conditions

Asymptotic Perturbation Theory: First Order

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}_1' = \boldsymbol{R}_1$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}'_\perp + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}'_\perp \\ 0 \end{pmatrix} \qquad \boldsymbol{R}_1 = \begin{pmatrix} -\left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \boldsymbol{u}'_{0\perp} - \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \boldsymbol{u}'_{0\perp} \right) \\ -\partial_Z \hat{p}'_0 + \hat{\vartheta}'_0 - A^2 Ro \left((\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \hat{w}'_0 - \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{w}'_0 \right) \\ -\partial_Z \hat{w}'_0 \\ -(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp)\, \hat{\vartheta}'_0 - \hat{w}'_0 \partial_Z \overline{T}_0 + \frac{1}{Pe} \left(\nabla_\perp^2 + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{\vartheta}'_0 \end{pmatrix}$$

- Find $\boldsymbol{v}^\dagger$ orthogonal to the span of $\mathcal{L}_{geo}\boldsymbol{v}'$. Then

$$\left\langle \mathcal{L}_{geo}^\dagger \boldsymbol{v}^\dagger, \boldsymbol{v}_1' \right\rangle_V = \left\langle \boldsymbol{v}^\dagger, \mathcal{L}_{geo}\boldsymbol{v}_1' \right\rangle_V = 0 \quad \Rightarrow \quad \left\langle \boldsymbol{v}^\dagger, \boldsymbol{R}_1 \right\rangle_V = 0$$

IBP

solvability conditions

Asymptotic Perturbation Theory: First Order

$$\mathcal{O}(Ro): \quad \mathcal{L}_{geo}\boldsymbol{v}_1' = \boldsymbol{R}_1$$

$$\mathcal{L}_{geo}\boldsymbol{v} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp' + \nabla_\perp \hat{p}' \\ 0 \\ \nabla_\perp \cdot \boldsymbol{u}_\perp' \\ 0 \end{pmatrix} \qquad \boldsymbol{R}_1 = \begin{pmatrix} -\left(\left(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp\right) \boldsymbol{u}_{0\perp}' - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \boldsymbol{u}_{0\perp}' \right) \\ -\partial_Z \hat{p}_0' + \hat{\vartheta}_0' - A^2 Ro \left(\left(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp\right) \hat{w}_0' - \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}_0' \right) \\ -\partial_Z \hat{w}_0' \\ -\left(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp\right) \hat{\vartheta}_0' - \hat{w}_0' \partial_Z \overline{T}_0 + \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}_0' \end{pmatrix}$$

- Find $\boldsymbol{v}^\dagger$ orthogonal to the span of $\mathcal{L}_{geo}\boldsymbol{v}'$. Then

$$\left\langle \mathcal{L}_{geo}^\dagger \boldsymbol{v}^\dagger, \boldsymbol{v}_1' \right\rangle_V = \left\langle \boldsymbol{v}^\dagger, \mathcal{L}_{geo}\boldsymbol{v}_1' \right\rangle_V = 0 \quad \Rightarrow \quad \left\langle \boldsymbol{v}^\dagger, \boldsymbol{R}_1 \right\rangle_V = 0$$

IBP

solvability conditions

$$\boxed{\mathcal{L}_{geo}^\dagger = -\mathcal{L}_{geo}, \quad \boldsymbol{v}^\dagger \propto \boldsymbol{v}_0'}$$

three solvability conditions

$$\boldsymbol{v}^\dagger = \begin{pmatrix} -\partial_y \psi^\dagger \\ \partial_y \psi^\dagger \\ 0 \\ \psi^\dagger \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w^\dagger \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vartheta^\dagger \end{pmatrix}$$

Reduced Quasi-Geostrophic Equations (Unified across scales)

Balance: $p = \psi'_0, \quad \boldsymbol{u} = (-\partial_y \psi'_0, \partial_x \psi'_0, \hat{w}'_0), \quad \zeta'_0 = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}'_0 = \nabla^2_\perp \psi'_0, \qquad T = \overline{T}(Z) + Ro\, \hat{\vartheta}'_0.$

Fluctuating Equations:

Vert. Vorticity

$$\left(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp\right) \zeta'_0 - \partial_Z \hat{w}'_0 = \frac{1}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \zeta'_0$$

Vert. Velocity

$$(ARo)^2 (\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp) \hat{w}'_0 + \partial_Z \psi'_0 = \widetilde{\Gamma} \hat{\vartheta}'_0 + \frac{(ARo)^2}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{w}'_0$$

Temp. Fluct.

$$(\partial_t + \boldsymbol{u}'_{0\perp} \cdot \nabla_\perp) \hat{\vartheta}'_0 + \hat{w}'_0 \partial_Z \overline{T}_0 = \frac{1}{Pe} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{\vartheta}'_0$$

Mean Equations:

Hydrostatic balance

$$\partial_Z \overline{p}_0 = \widetilde{\Gamma} \overline{T}_0$$

Mean temp

$$\frac{1}{Ro^2} \partial_t \overline{T} + \partial_Z (\overline{\hat{w}'_0 \hat{\vartheta}'_0}) = \frac{1}{(ARo)^2 Pe} \partial_{ZZ} \overline{T}_0$$

Reduced Quasi-Geostrophic Equations (Unified across scales)

Balance: $p = \psi_0', \quad \boldsymbol{u} = (-\partial_y \psi_0', \partial_x \psi_0', \hat{w}_0'), \quad \zeta_0' = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_0' = \nabla_\perp^2 \psi_0', \quad T = \overline{T}(Z) + Ro\,\hat{\vartheta}_0'.$

Fluctuating Equations:

Vert. Vorticity

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\, \zeta_0' - \partial_Z \hat{w}_0' = \frac{1}{Re}\left(\nabla_\perp \cdot + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \zeta_0'$$

Vert. Velocity

$$(ARo)^2(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\hat{w}_0' + \partial_Z \psi_0' = \widetilde{\Gamma}\hat{\vartheta}_0' + \frac{(ARo)^2}{Re}\left(\nabla_\perp \cdot + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{w}_0'$$

Temp. Fluct.

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\hat{\vartheta}_0' + \hat{w}_0'\partial_Z \overline{T}_0 = \frac{1}{Pe}\left(\nabla_\perp \cdot + \frac{1}{A}\hat{\boldsymbol{z}}\partial_Z\right)^2 \hat{\vartheta}_0'$$

- Dispersion relation inertial waves $\widetilde{\Gamma} = 0$:

$$\omega_{wave}^2 = \frac{1}{Ro^2}\left(\frac{k_z^2}{A^2 k_\perp^2 + k_z^2}\right), \qquad \lim_{A=o(Ro^{-1})} = Ro^{-2}, \quad \lim_{A=Ro^{-1}} = \frac{k_z^2}{k_\perp^2} \qquad \omega_{eddy} = 0$$

all i.w's filtered — slow i.w's retained

- Slow inertial waves are retained in non-hydrostatic QG flows.

Dispersion Relation

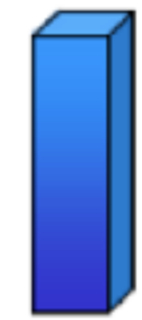
$$\Gamma = \frac{A}{Fr_H^2} = \frac{1}{ARo^2}\widetilde{\Gamma}, \quad \widetilde{\Gamma} = \frac{1}{\widetilde{Fr}_H^{\,2}} = \mathcal{O}(1)$$

- Unstratified case: $\widetilde{\Gamma} = 0$

$$\omega^2_{wave} = \frac{1}{Ro^2}\left(\frac{k_z^2}{A^2 k_\perp^2 + k_z^2}\right)$$

$$\lim_{A=o(1)} \omega = Ro^{-1} \to \infty$$

$$\lim_{A=Ro^{-1}} \omega = \pm\frac{k_z}{k_\perp} = \mathcal{O}(1)$$



all i.w's are fast and filtered from reduced model. Classical QG

slow i.w's retained in reduced model

Slow waves interact w geostrophic mode

Non hydrostatic QG

- Stratified case: $\widetilde{\Gamma} \neq 0$

$$\omega^2_{wave} = \frac{1}{Ro^2}\left(\frac{1}{A^2 k_\perp^2 + k_z^2}\right)\left(k_z^2 + \widetilde{\Gamma}\partial_Z \overline{T}\, k_\perp^2\right)$$

$$\lim_{A=o(1)} \omega^2_{wave} = \frac{1}{Ro^2}\left(1 + \widetilde{\Gamma}\partial_Z \overline{T}\,\frac{k_\perp^2}{k_z^2}\right) = \mathcal{O}(Ro^{-2}) \to \infty$$

$$\lim_{A=o(1)} \omega^2_{wave} = \left(\frac{k_z^2}{k_\perp^2} + \widetilde{\Gamma}\partial_Z \overline{T}\right) = \mathcal{O}(1)$$

Dispersion Relation (anisotropically scaled)

$$\omega^2_{wave} = \frac{1}{Ro^2}\left(\frac{1}{A^2 k_\perp^2 + k_z^2}\right)\left(k_z^2 + \widetilde{\Gamma}\partial_Z \overline{T}\, k_\perp^2\right)$$

- Stratification Parameter $\quad \Gamma = \frac{A}{Fr_H^2} = \frac{1}{ARo^2}\widetilde{\Gamma}, \quad \widetilde{\Gamma} = \frac{1}{\widetilde{Fr}_H^{\,2}} = \mathcal{O}(1)$

$\widetilde{\Gamma} = 0$ Unstratified

$\widetilde{\Gamma} \neq 0$ Stratified case

Classical QG

$\lim_{A=o(1)}$

$$\omega = Ro^{-1} \to \infty$$

$$\omega^2_{wave} = \frac{1}{Ro^2}\left(1 + \widetilde{\Gamma}\partial_Z \overline{T}\,\frac{k_\perp^2}{k_z^2}\right) = \mathcal{O}(Ro^{-2}) \to \infty$$

all i.w's are fast and filtered from reduced model.

Nonhydrostatic QG

$\lim_{A=o(1)}$

$$\omega = \pm\frac{k_z}{k_\perp} = \mathcal{O}(1)$$

$$\omega^2_{wave} = \left(\frac{k_z^2}{k_\perp^2} + \widetilde{\Gamma}\partial_Z \overline{T}\right) = \mathcal{O}(1)$$

slow igw's retained in reduced model.

Slow waves interact w geostrophic mode.

Reduced Quasi-Geostrophic Equations (Unified across scales)

Balance: $p = \psi_0', \quad \boldsymbol{u} = (-\partial_y \psi_0', \partial_x \psi_0', \hat{w}_0'), \quad \zeta_0' = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_0' = \nabla_\perp^2 \psi_0', \quad T = \overline{T}(Z) + Ro\, \hat{\vartheta}_0'.$

Fluctuating Equations:

Vert. Vorticity

$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp)\, \zeta_0' - \partial_Z \hat{w}_0' = \frac{1}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \zeta_0'$$

Vert. Velocity

$$(ARo)^2 (\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{w}_0' + \partial_Z \psi_0' = \widetilde{\Gamma} \hat{\vartheta}_0' + \frac{(ARo)^2}{Re} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{w}_0'$$

Temp. Fluct.

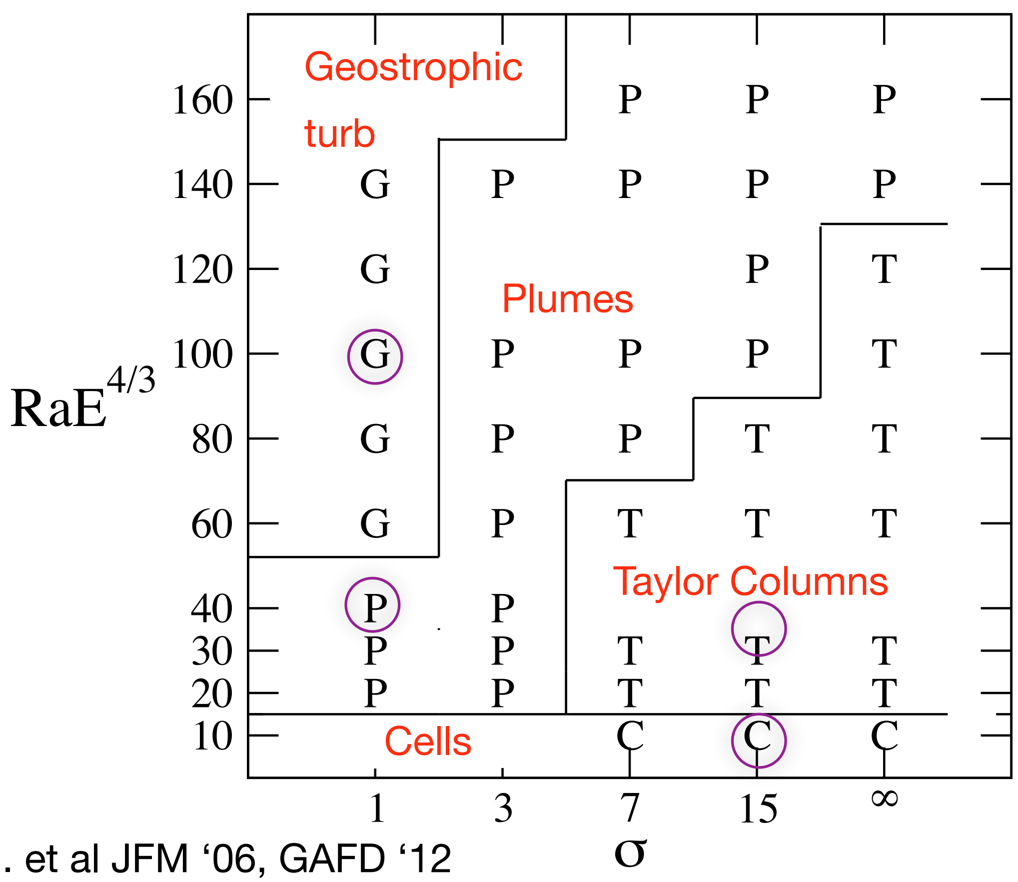
$$(\partial_t + \boldsymbol{u}_{0\perp}' \cdot \nabla_\perp) \hat{\vartheta}_0' + \hat{w}_0' \partial_Z \overline{T}_0 = \frac{1}{Pe} \left(\nabla_\perp . + \frac{1}{A} \hat{\boldsymbol{z}} \partial_Z \right)^2 \hat{\vartheta}_0'$$

- Conservation Laws for non-hydrostatic QG

$$E = \frac{1}{2} \left\langle |\nabla^\perp \psi|^2 + w^2 + \frac{1}{Fr^2} \frac{\theta^2}{\partial_Z \overline{T}} \right\rangle_V , \qquad Q_{PV} = \left(\zeta + \frac{1}{Fr^2} \partial_Z \left(\frac{\theta}{\partial_Z \overline{T}} \right) \right) + J_\perp(w, \theta)$$

Quasi-Geostrophic RBC Flow Regimes

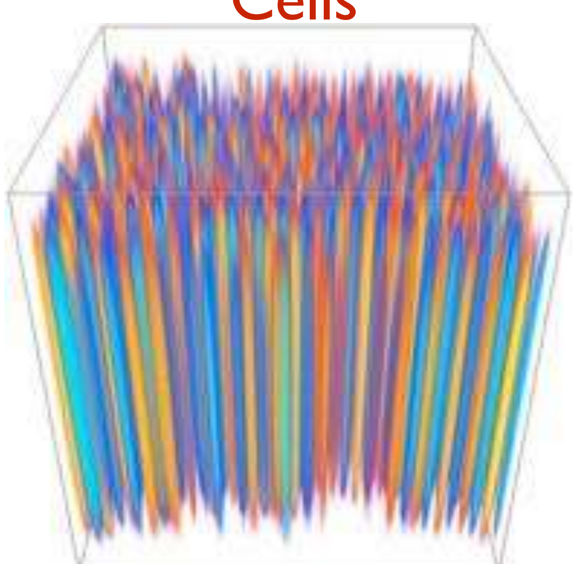
Two new regimes



J. et al JFM '06, GAFD '12

$RaE^{4/3} = 10, \sigma = 15$

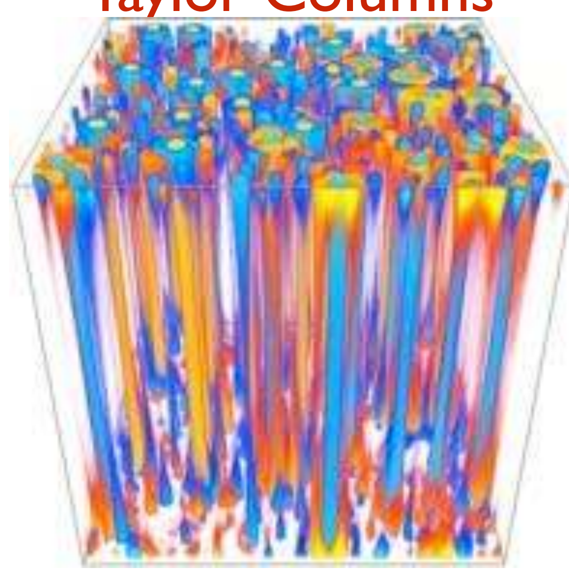
Cells



KJ. & Knobloch PoF 99, JMP '07

$RaE^{4/3} = 40, \sigma = 15$

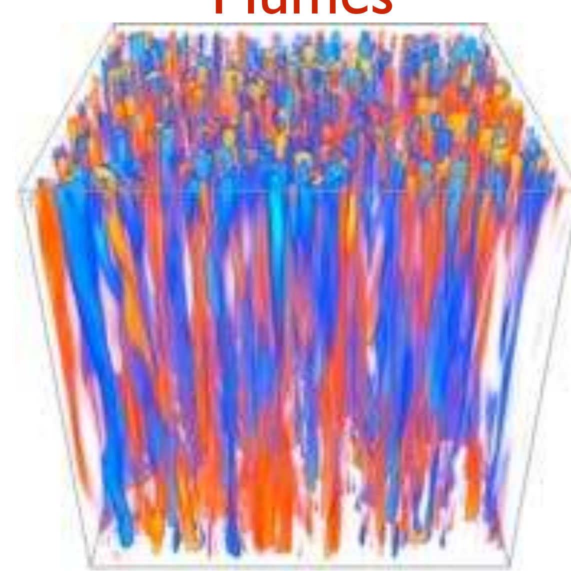
Taylor Columns



Grooms,, KJ et al PRL '10

$RaE^{4/3} = 40, \sigma = 1$

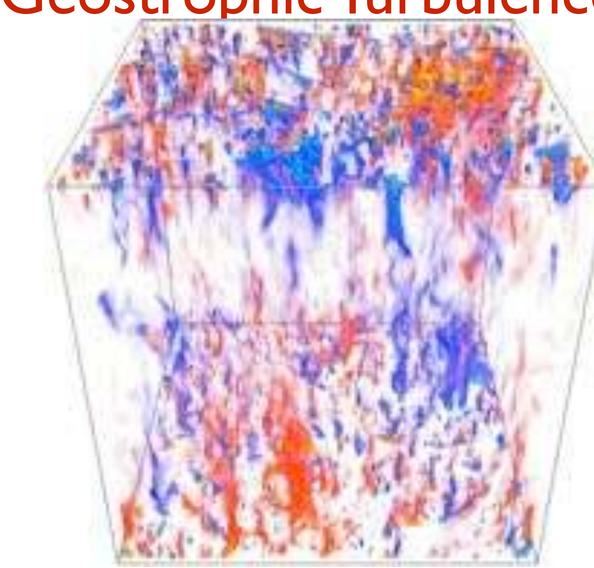
Plumes



KJ et al GAFD '12

$RaE^{4/3} = 100, \sigma = 1$

Geostrophic Turbulence



2014 - Rubio, KJ, et al PRL; Stellmach, KJ, et al PRL

Favier et al PoF; Guervilly et al JFM

Summary and Outlook

- H-QG and NH-QG model may viewed in a unified way

  - Aspect Ratio $A=H/L$ enters as the control parameter

  - Universal velocity scaling identified $W = A\, Ro$

- NH-QG permitted an extensive survey of QG Rayleigh-Benard Convection

- Effort requires synergy between Lab exp's, DNS, and reduced modeling

- Outlook

  - Dynamo problem, generation of magnetic fields

  - LSV & Inverse Cascade

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