Turbulence of gravitational waves in the primordial universe (wave turbulence)

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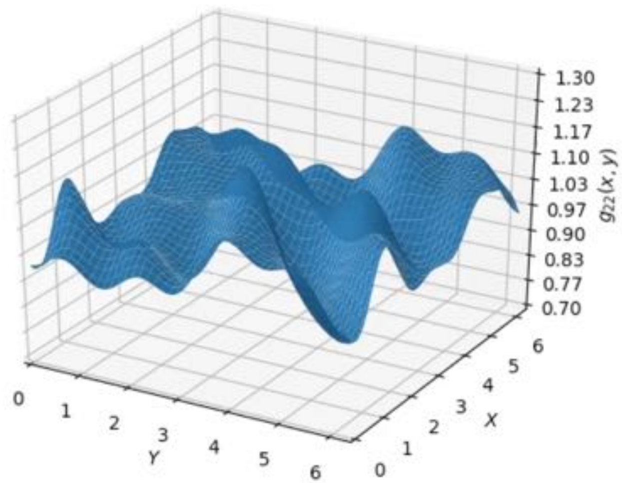

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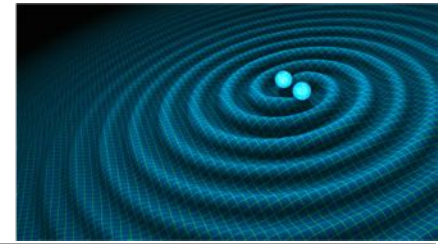
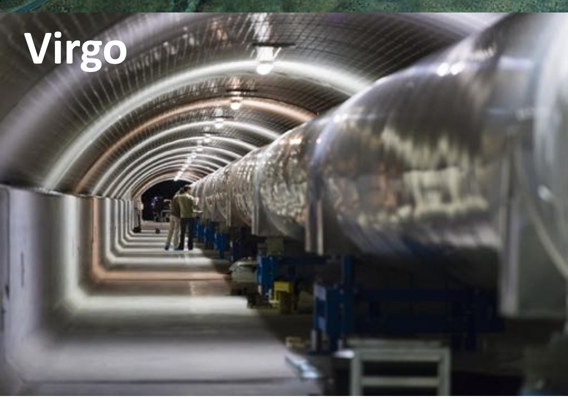
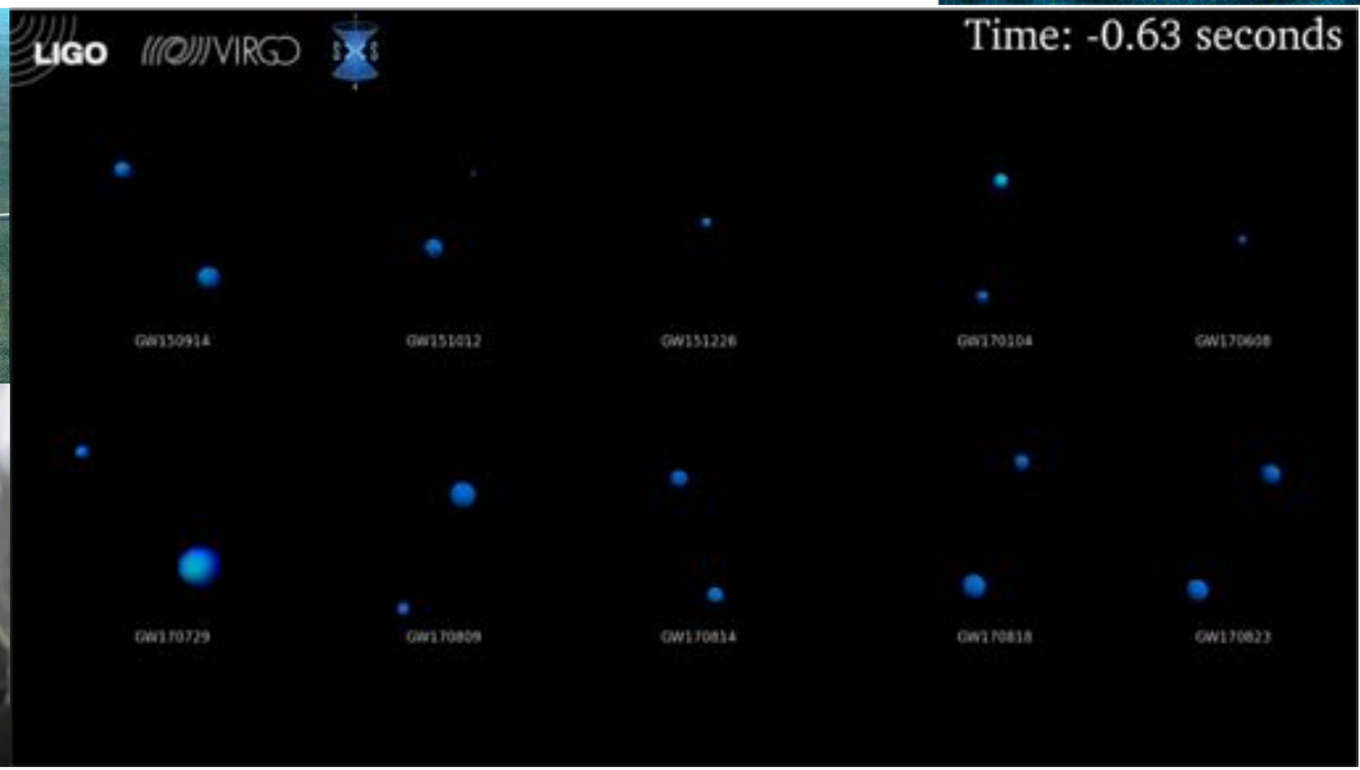
Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

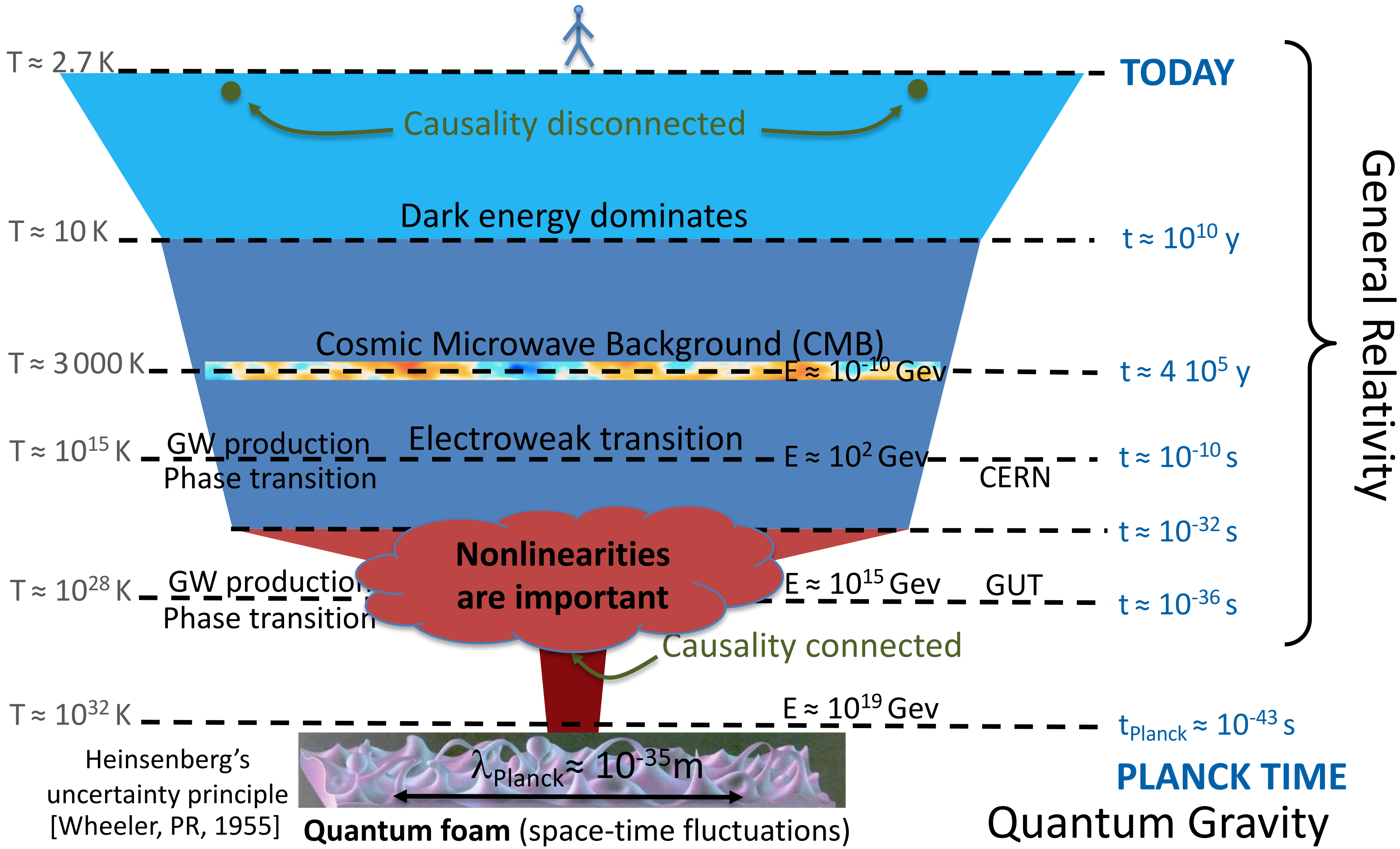






New detections [see arXiv:1811.12907v1]

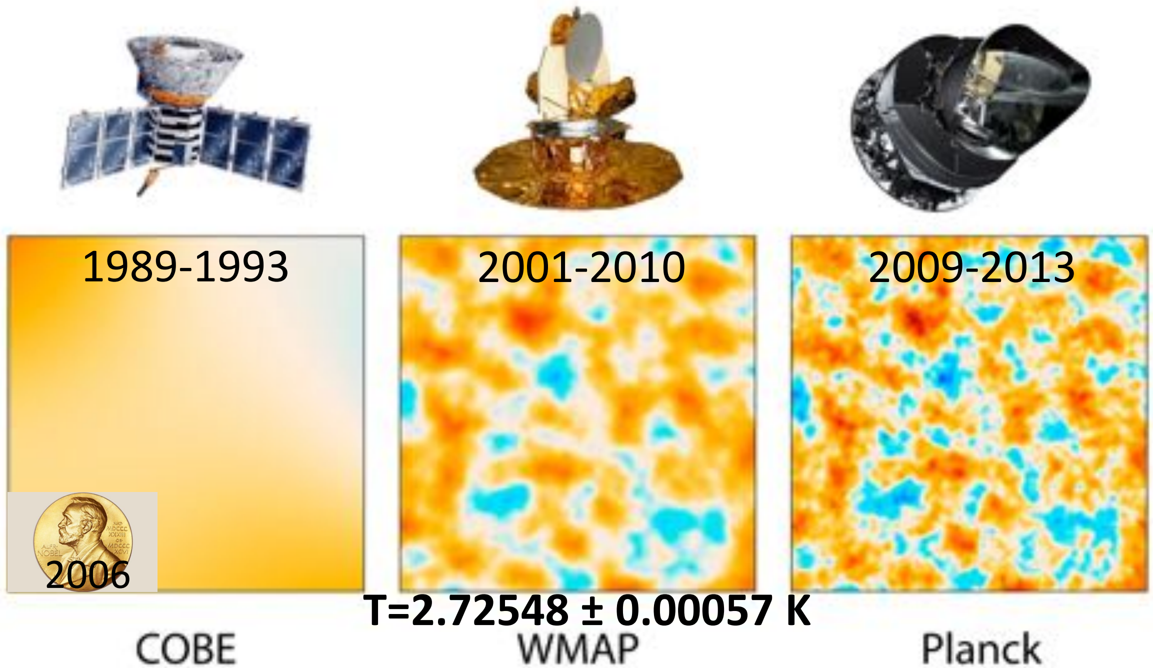
History of the Universe



Inflation is explained by introducing an hypothetical scalar field called *inflaton* [Guth, 1981]

The Cosmic Microwave Background (1964) is statistically **uniform**

➔ **horizon** (causal) problem is solved with inflation



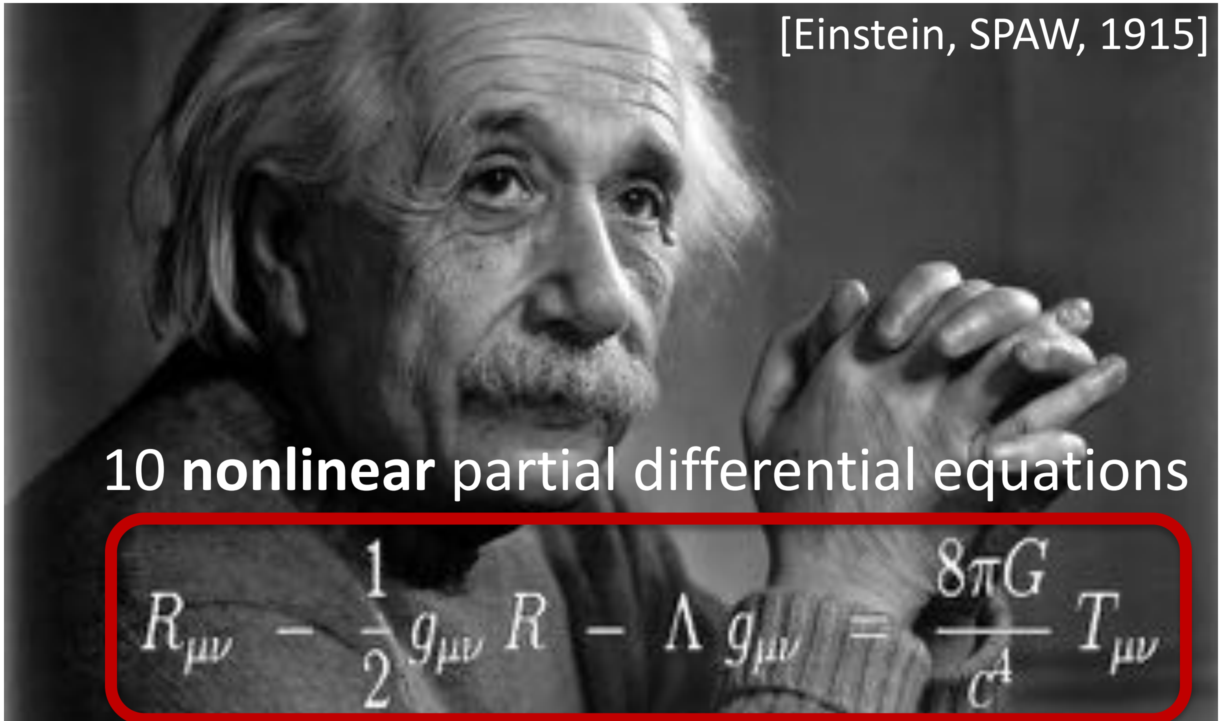
Homogeneous up to 1/10 000

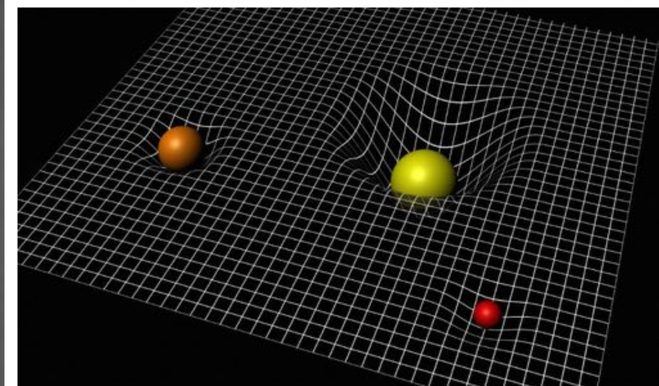
"So far, the details of inflation are unknown, and the whole idea of inflation remains a speculation, though one that is increasingly plausible." Weinberg, Cosmology, 2008.

[Ijjas+, PLB, 2013; 2014]

We need a convincing **physical** mechanism for inflation

Einstein equations





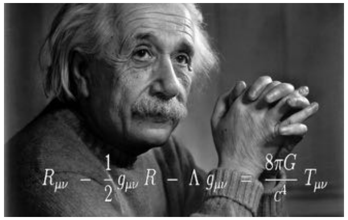
$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left[\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk} \right]$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

- $R_{\mu\nu}$: Ricci tensor
- R : Ricci scalar
- $g_{\mu\nu}$: metric tensor
- Λ : cosmological constant

- $\Gamma^\alpha_{\mu\nu}$: Christoffel symbol
- $T_{\mu\nu}$: stress-energy
- $G = 6.67\ 10^{-11}\ m^3 kg^{-1} s^{-2}$
- $c = 2.99\ 10^8\ ms^{-1}$



Gravitational waves

$\Lambda = 0$

Exact linear solutions in an empty – flat – Universe: $\boxed{R_{\mu\nu} = 0}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

Poincaré-Minkowski space-time metric

Effect of a + gravitational wave on a ring of particles (h = 0.5)

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

$$h^{+}_{\mu\nu} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Wave Turbulence

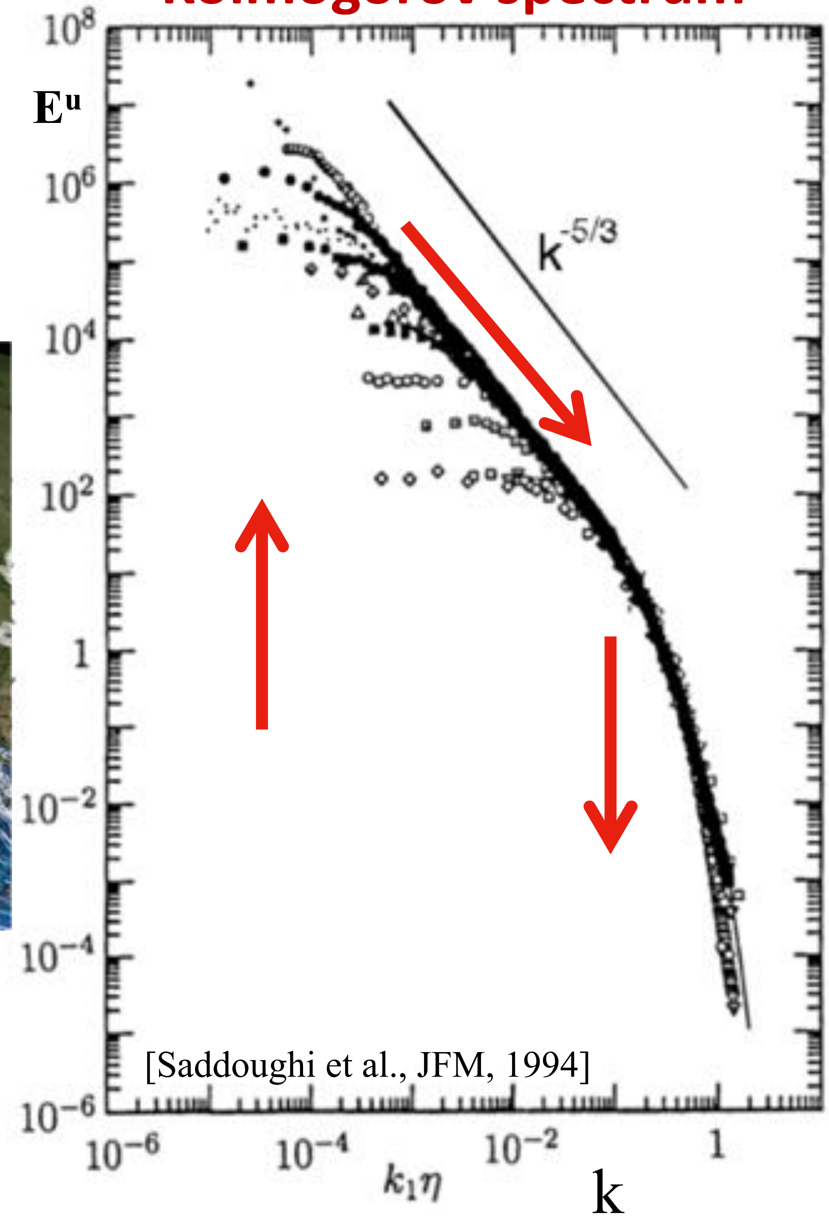
Strong (eddy) turbulence

Ocean current flows /eddies

(Feb. 2005 to Jan. 2006)

[G. Shirah, NASA/GSFC]

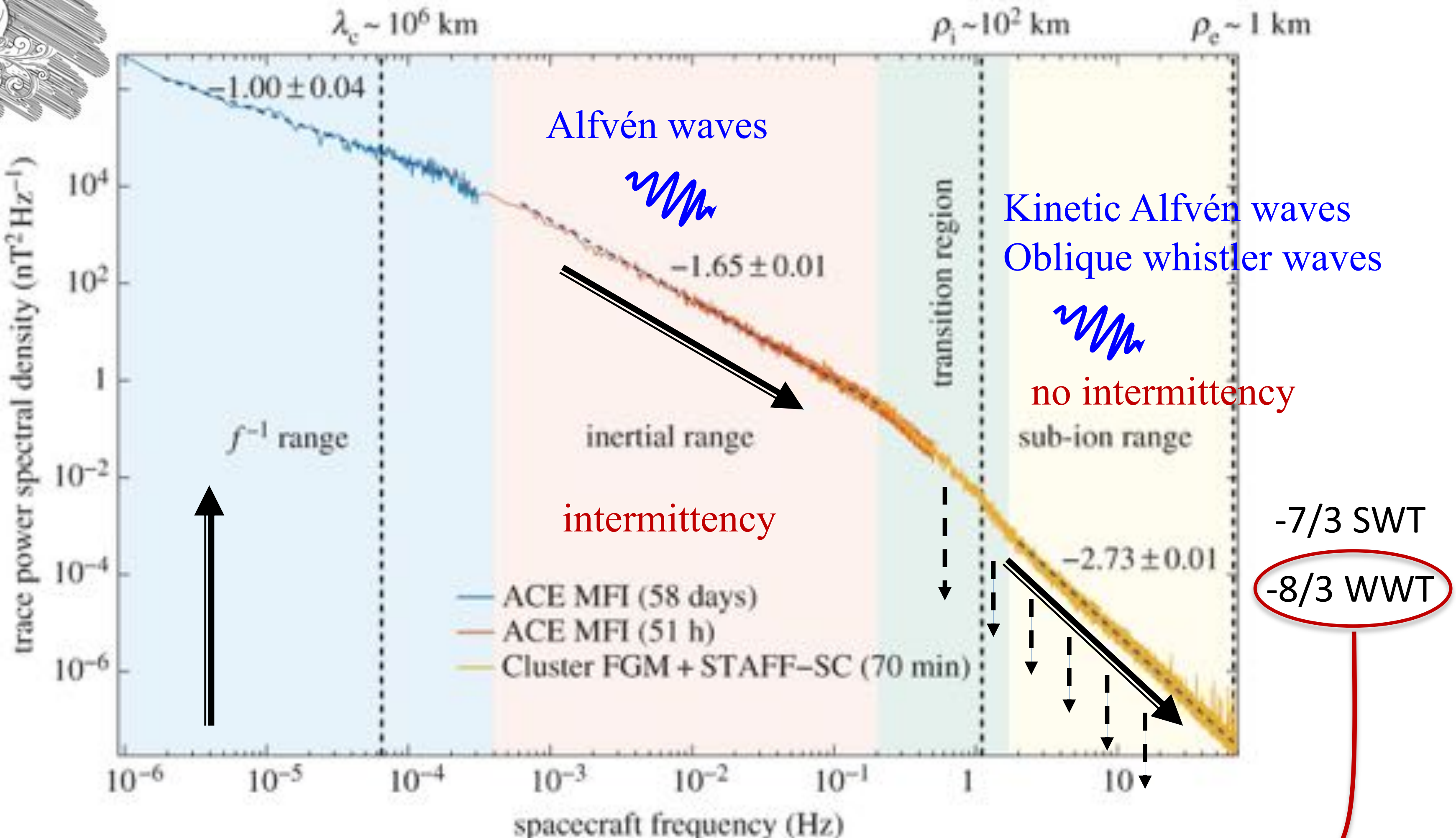
Kolmogorov spectrum



Strong & Weak Wave turbulence

Solar wind
(~ 1 AU)

8 decades !!

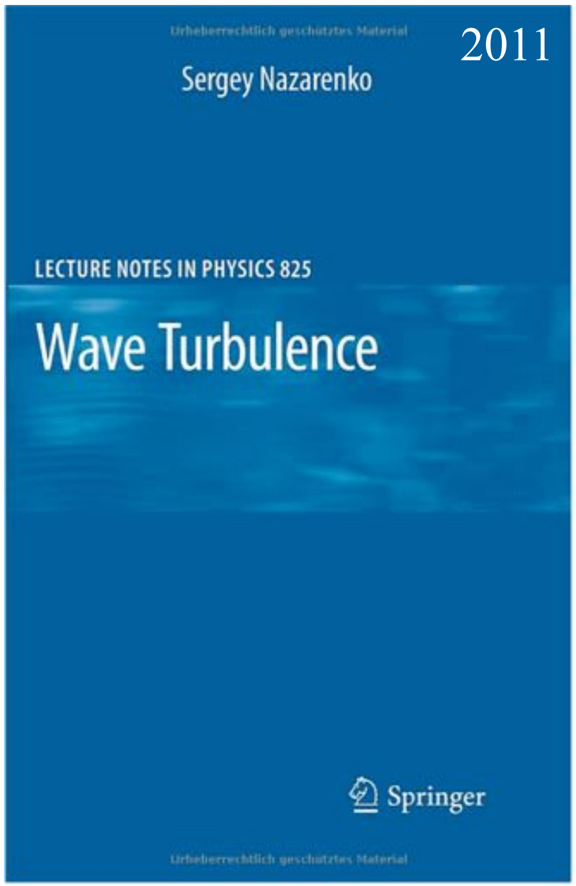


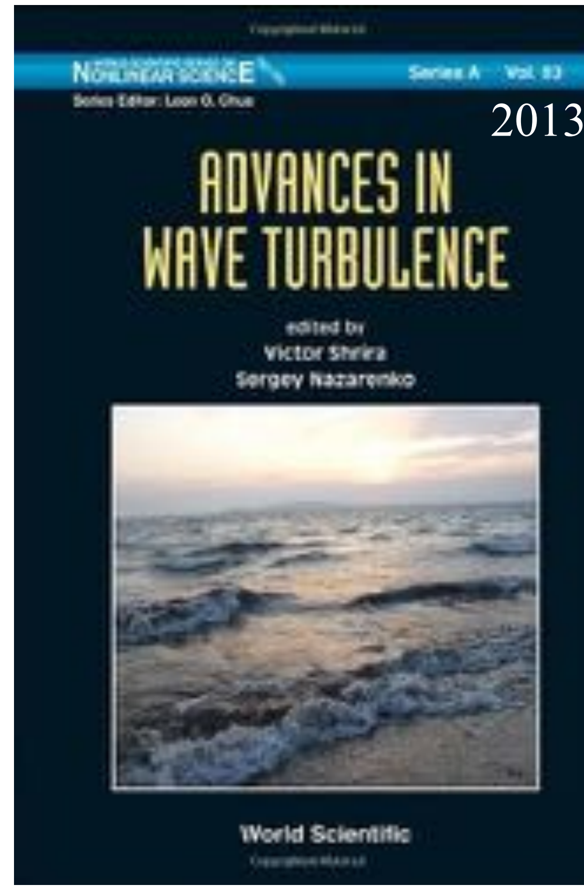
[Kiyani+, Phil. Trans. R. Soc. A, 2015]

[David & SG, 2019, in press]

Weak wave turbulence

Statistical theory of **weakly nonlinear** waves





For magnetized plasmas see : SG, Nonlin. Proc. Geophys. (2009)

Weak wave turbulence theory

+ Natural **asymptotical closure** of the hierarchy of moment equations
  [Benney & Saffman, PRSLA, 1966; Benney & Newell, JMP, 1967]

+ The kinetic equations admits **stationary finite flux** solutions
  [Zakharov & Filonenko, DAN, 1966; Kraichnan, PoF, 1967]

– Finite flux spectra not valid **over all k's** $\rightarrow$ strong turbulence
  [Galtier+, JPP, 2000; Meyrand+, PRL, 2016]

– Experiments and dns show **some limitations** in the predictions
  [Morize+, PoF, 2005; Nazarenko, NJP, 2007]

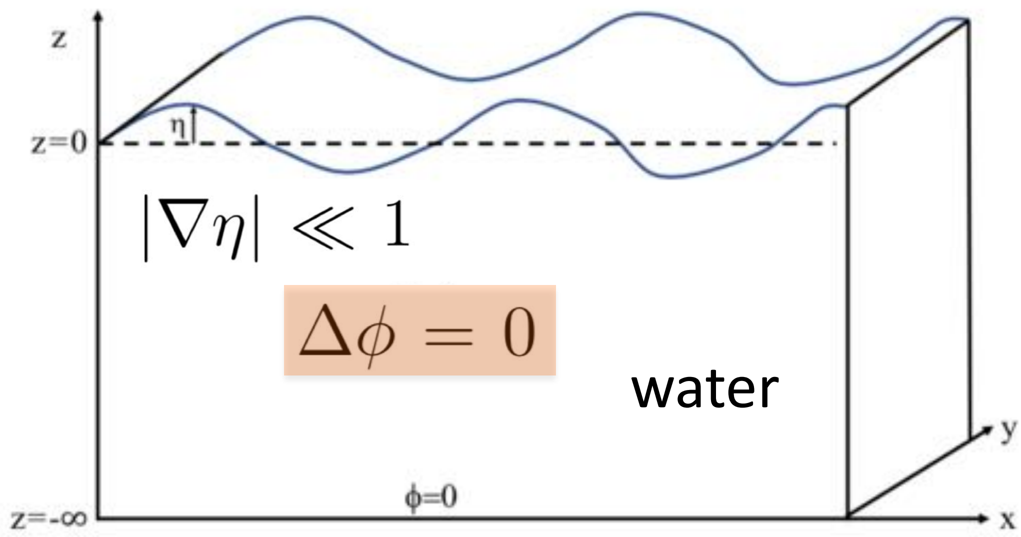
Capillary wave turbulence

$\nabla \cdot \mathbf{u} = 0$ incompressible

$\mathbf{u} = \nabla\phi$ irrotational

Φ(x,y,z,t): velocity potential

η(x,y,t): free surface



$|\nabla\eta| \ll 1$

$\Delta\phi = 0$

water

[Zakharov & Filonenko, JAMTP, 1967]

$$\frac{d\eta}{dt} = u_z = \frac{\partial\phi}{\partial z}|_\eta$$

$$\frac{\partial\phi}{\partial t}|_\eta - \frac{\partial\phi}{\partial z}|_\eta\frac{\partial\eta}{\partial t} = -\frac{1}{2}(\nabla\phi)^2|_\eta + \sigma\Delta\eta$$

with: $\sigma = \gamma/\rho_{water}$

γ: surface tension (=0.07N/m)

$$\Rightarrow \quad \begin{aligned} \frac{\partial\eta}{\partial t} &= -\nabla_\perp\phi|_0 \cdot \nabla_\perp\eta + \frac{\partial\phi}{\partial z}|_0 + \eta\frac{\partial^2\phi}{\partial z^2}|_0 , \\ \frac{\partial\phi}{\partial t}|_0 + \eta\frac{\partial^2\phi}{\partial t\partial z}|_0 &= -\frac{1}{2}(\nabla\phi)^2|_0 + \sigma\Delta\eta . \end{aligned}$$

$k > k_*$

$k_* \equiv \sqrt{g\rho_{eau}/\gamma}$

Capillary wave turbulence

$$\frac{\partial \hat{\eta}_k}{\partial t} - k\hat{\phi}_k = \int [(\mathbf{p}\cdot\mathbf{q})\hat{\phi}_p\hat{\eta}_q + p^2\hat{\phi}_p\hat{\eta}_q]\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\,, \quad (1.14)$$

$$\frac{\partial \hat{\phi}_k}{\partial t} + \sigma k^2\hat{\eta}_k = \frac{1}{2}\int [(\mathbf{p}\cdot\mathbf{q} - pq)\hat{\phi}_p\hat{\phi}_q + 2\sigma p^3\hat{\eta}_p\hat{\eta}_q]\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \quad (1.15)$$

Canonical variables:

$$\hat{\eta}_k \equiv \left(\frac{4}{\sigma k}\right)^{1/4}\sum_s A_k^s\,,$$

$$\hat{\phi}_k \equiv -i(4\sigma k)^{1/4}\sum_s sA_k^s\,,$$

$s = \pm$

$\omega_k = \sqrt{\sigma k^3}$

$$\frac{\partial A_k^s}{\partial t} + is\omega_k A_k^s = -\frac{i\sigma^{1/4}}{2\sqrt{2}}\int\sum_{s_p s_q} s_p s_q A_p^{s_p}A_q^{s_q}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \quad (1.24)$$

$$\Rightarrow \left[s(\mathbf{p}\cdot\mathbf{q}+pq)\left(\frac{pq}{k}\right)^{1/4} + s_q(\mathbf{k}\cdot\mathbf{p}-kp)\left(\frac{pk}{q}\right)^{1/4} + s_p(\mathbf{k}\cdot\mathbf{q}-kq)\left(\frac{qk}{p}\right)^{1/4}\right]d\mathbf{p}d\mathbf{q}$$

$$-\frac{is\sigma^{1/4}}{2\sqrt{2}}\int\sum_{s_p s_q}\frac{(s_p\omega_p+s_q\omega_q)(s\omega_k - s_p\omega_p - s_q\omega_q)}{\sigma(kpq)^{1/4}}A_p^{s_p}A_q^{s_q}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\,.$$

Capillary wave turbulence

$$A_k^s \equiv \epsilon a_k^s e^{-is\omega_k t} \qquad \epsilon \ll 1$$

$$\Rightarrow \quad \frac{\partial a_k^s}{\partial t} = i\epsilon \int \sum_{s_p s_q} L_{-kpq}^{-s s_p s_q} a_p^{s_p} a_q^{s_q} e^{i\Omega_{k,pq}t} \delta_{k,pq} d\mathbf{p} d\mathbf{q} \,,$$

$$\Omega_{k,pq} \equiv s\omega_k - s_p\omega_p - s_q\omega_q, \qquad \delta_{k,pq} \equiv \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})$$

$$L_{kpq}^{s s_p s_q} \equiv \frac{s_p s_q \sigma^{1/4}}{2\sqrt{2}} \left[s(\mathbf{p}\cdot\mathbf{q} + pq) \left(\frac{pq}{k}\right)^{1/4} + s_p(\mathbf{k}\cdot\mathbf{q} + kq)\left(\frac{qk}{p}\right)^{1/4} + s_q(\mathbf{k}\cdot\mathbf{p} + kp)\left(\frac{pk}{q}\right)^{1/4} \right] . \qquad (1.28)$$

$$\begin{aligned}
L_{kpq}^{s s_p s_q} &= L_{kqp}^{s s_q s_p} \,, \\
L_{0pq}^{s s_p s_q} &= 0 \,, \\
L_{-k-p-q}^{s s_p s_q} &= L_{kpq}^{s s_p s_q} \,, \\
L_{kpq}^{-s -s_p -s_q} &= -L_{kpq}^{s s_p s_q} \,, \\
s s_q L_{qpk}^{s_q s_p s} &= L_{kpq}^{s s_p s_q} \,, \\
s s_p L_{pkq}^{s_p s s_q} &= L_{kpq}^{s s_p s_q} \,.
\end{aligned}$$

[Zakharov & Filonenko, JAMTP, 1967]

⇒ We are ready for a statistical development

Capillary wave (homogeneous) turbulence

$$
\begin{aligned}
\langle a_k^s a_{k'}^{s'} \rangle &= q_{kk'}^{ss'}(\mathbf{k},\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')\,, \\
\langle a_k^s a_{k'}^{s'} a_{k''}^{s''} \rangle &= q_{kk'k''}^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\,, \\
\langle a_k^s a_{k'}^{s'} a_{k''}^{s''} a_{k'''}^{s'''} \rangle &= q_{kk'k''k'''}^{ss's''s'''}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''') \\
&+ q_{kk'}^{ss'}(\mathbf{k},\mathbf{k}')q_{k''k'''}^{s''s'''}(\mathbf{k}'',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}')\delta(\mathbf{k}''+\mathbf{k}''') \\
&+ q_{kk''}^{ss''}(\mathbf{k},\mathbf{k}'')q_{k'k'''}^{s's'''}(\mathbf{k}',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}'')\delta(\mathbf{k}'+\mathbf{k}''') \\
&+ q_{kk'''}^{ss'''}(\mathbf{k},\mathbf{k}''')q_{k'k''}^{s's''}(\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}''')\delta(\mathbf{k}'+\mathbf{k}'')
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \langle a_k^s a_{k'}^{s'} \rangle}{\partial t} &= \left\langle \frac{\partial a_k^s}{\partial t} a_{k'}^{s'} \right\rangle + \left\langle a_k^s \frac{\partial a_{k'}^{s'}}{\partial t} \right\rangle \\
&= i\epsilon \int \sum_{s_p s_q} L_{-kpq}^{-s s_p s_q} \langle a_{k'}^{s'} a_p^{s_p} a_q^{s_q} \rangle e^{i\Omega_{k,pq}t}\delta_{k,pq}d\mathbf{p}d\mathbf{q} \\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k'pq}^{-s' s_p s_q} \langle a_k^s a_p^{s_p} a_q^{s_q} \rangle e^{i\Omega_{k',pq}t}\delta_{k',pq}d\mathbf{p}d\mathbf{q}\,.
\end{aligned}
$$

Asymptotic closure:
only resonance terms survive

$$\langle A_k^s A_{k'}^{s'} \rangle = \epsilon^2 \langle a_k^s a_{k'}^{s'} \rangle \exp(-i(s\omega_k + s'\omega_{k'})t)$$

$$\Delta(\Omega_{kk'k''}) = \int_0^{t\gg 1/\omega} e^{i\Omega_{kk'k''}t'}dt' = \frac{e^{i\Omega_{kk'k''}t}-1}{i\Omega_{kk'k''}}$$

$$\Delta(x) \to \pi\delta(x) + i\mathcal{P}(1/x)$$

$$
\begin{aligned}
\frac{\partial \langle a_k^s a_{k'}^{s'} a_{k''}^{s''} \rangle}{\partial t} &= \left\langle \frac{\partial a_k^s}{\partial t} a_{k'}^{s'} a_{k''}^{s''} \right\rangle + \left\langle a_k^s \frac{\partial a_{k'}^{s'}}{\partial t} a_{k''}^{s''} \right\rangle + \left\langle a_k^s a_{k'}^{s'} \frac{\partial a_{k''}^{s''}}{\partial t} \right\rangle \\
&= i\epsilon \int \sum_{s_p s_q} L_{-kpq}^{-s s_p s_q} \langle a_{k'}^{s'} a_{k''}^{s''} a_p^{s_p} a_q^{s_q} \rangle e^{i\Omega_{k,pq}t}\delta_{k,pq}d\mathbf{p}d\mathbf{q} \\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k'pq}^{-s' s_p s_q} \langle a_k^s a_{k''}^{s''} a_p^{s_p} a_q^{s_q} \rangle e^{i\Omega_{k',pq}t}\delta_{k',pq}d\mathbf{p}d\mathbf{q} \\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k''pq}^{-s'' s_p s_q} \langle a_k^s a_{k'}^{s'} a_p^{s_p} a_q^{s_q} \rangle e^{i\Omega_{k'',pq}t}\delta_{k'',pq}d\mathbf{p}d\mathbf{q}
\end{aligned}
$$

$$
\begin{aligned}
& q_{kk'k''}^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') = i\epsilon\Delta(\Omega_{kk'k''})\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') \\
& \Big\{ \left[L_{-k-k'-k''}^{-s-s'-s''} + L_{-k-k''-k'}^{-s-s''-s'} \right] q_{k''-k''}^{s''-s''}(\mathbf{k}'',-\mathbf{k}'')q_{k'-k'}^{s'-s'}(\mathbf{k}',-\mathbf{k}') \\
& + \left[L_{-k'-k-k''}^{-s'-s-s''} + L_{-k'-k''-k}^{-s'-s''-s} \right] q_{k''-k''}^{s''-s''}(\mathbf{k}'',-\mathbf{k}'')q_{k-k}^{s-s}(\mathbf{k},-\mathbf{k}) \\
& + \left[L_{-k''-k'-k}^{-s''-s'-s} + L_{-k''-k-k'}^{-s''-s-s'} \right] q_{k-k}^{s-s}(\mathbf{k},-\mathbf{k})q_{k'-k'}^{s'-s'}(\mathbf{k}',-\mathbf{k}') \Big\}
\end{aligned}
$$

[Benney & Saffman, PRSLA, 1966;
Benney & Newell, JMP, 1967]

Capillary wave (homogeneous) turbulence

Kinetic equation

[Zakharov & Filonenko, JAMTP, 1967]

$$\frac{\partial q_k^s(\mathbf{k})}{\partial t} = 4\pi\epsilon^2 \int \sum_{s_p s_q} |L_{kpq}^{s s_p s_q}|^2 \delta(s\omega_k + s_p\omega_p + s_q\omega_q)\delta(\mathbf{k}+\mathbf{p}+\mathbf{q}) \quad (1.46)$$

$$s_p s_q \left[s_p s_q q_q^{s_q}(\mathbf{q}) q_p^{s_p}(\mathbf{p}) + s s_q q_q^{s_q}(\mathbf{q}) q_k^s(\mathbf{k}) + s s_p q_k^s(\mathbf{k}) q_p^{s_p}(\mathbf{p}) \right] d\mathbf{p} d\mathbf{q} \,,$$

$$|L_{kpq}^{s s_p s_q}|^2 \equiv \frac{\sqrt{\sigma}}{8} \left[s(\mathbf{p}\cdot\mathbf{q} + pq)\left(\frac{pq}{k}\right)^{1/4} + s_p(\mathbf{k}\cdot\mathbf{q} + kq)\left(\frac{qk}{p}\right)^{1/4} \right.$$

$$\left. + s_q(\mathbf{k}\cdot\mathbf{p} + kp)\left(\frac{pk}{q}\right)^{1/4} \right]^2 . \quad (1.47)$$

Resonant condition:

$$\begin{aligned} s\omega_k + s_p\omega_p + s_q\omega_q &= 0 \,, \\ \mathbf{k}+\mathbf{p}+\mathbf{q} &= 0 \,. \end{aligned}$$

Capillary wave (homogeneous) turbulence

Polarized energy spectrum: $e^s(\mathbf{k}) \equiv \omega_k q_k^s(\mathbf{k})$

$$\frac{\partial \int \sum_s e^s(\mathbf{k}) d\mathbf{k}}{\partial t} = \tag{1.52}$$

$$\frac{\pi\epsilon^2}{6\sigma} \int \sum_{s s_p s_q} |\tilde{L}_{kpq}^{s s_p s_q}|^2 \delta(s\omega_k + s_p\omega_p + s_q\omega_q) \delta(\mathbf{k}+\mathbf{p}+\mathbf{q})$$

$$(s\omega_k + s_p\omega_p + s_q\omega_q)\left[\frac{s\omega_k}{e^s(\mathbf{k})} + \frac{s_p\omega_p}{e^{s_p}(\mathbf{p})} + \frac{s_q\omega_q}{e^s(\mathbf{q})}\right] e^s(\mathbf{k}) e^{s_p}(\mathbf{p}) e^{s_q}(\mathbf{q}) d\mathbf{k} d\mathbf{p} d\mathbf{q}$$

$$= 0\,.$$

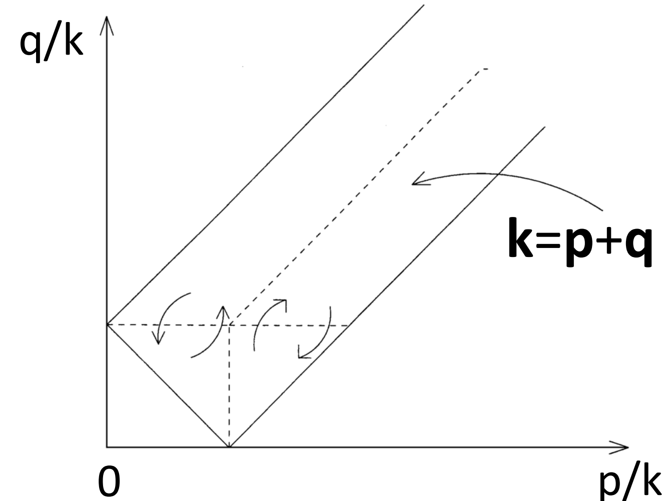
⇒ Detailed energy conservation on the resonant manifold

Exact solutions of the kinetic equations

Zakharov transformation: [Zakharov & Filonenko, DAN, 1966; Kraichnan, PoF, 1967] [Kuznetsov, JETP, 1972]

Two solutions for the energy spectrum

- zero-flux (thermodynamic)
- constant-flux (Kolmogorov-Zakharov)



For three waves

1D isotropic (constant flux) spectra:

$$E(k) = C\,P^{1/2}\,k^{-7/4}$$

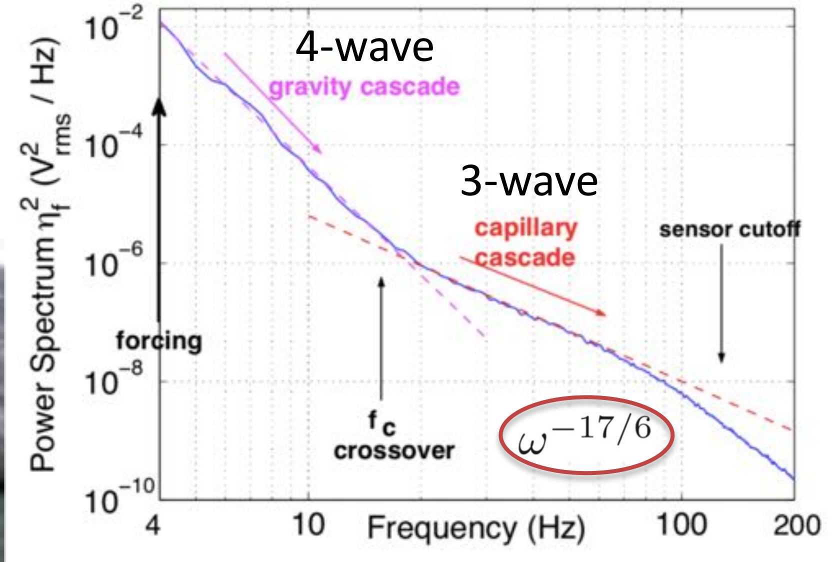
Direct cascade

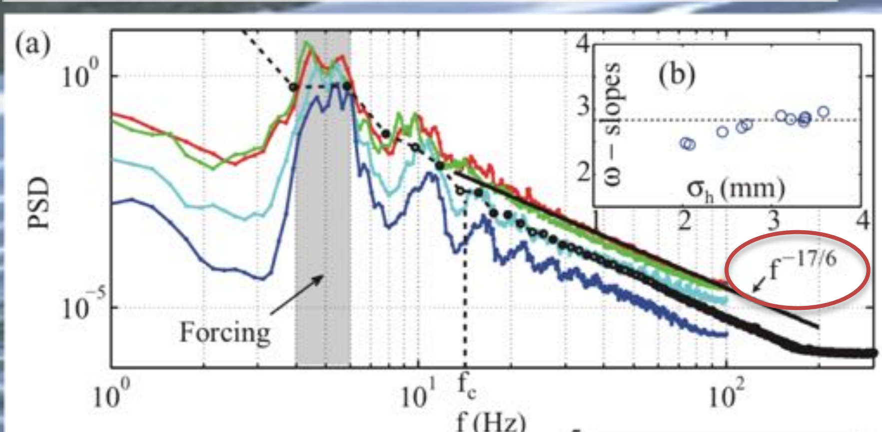
$$E(\omega) \sim \omega^{-7/4} \qquad E^{\eta}(\omega) \sim \omega^{-17/6}$$

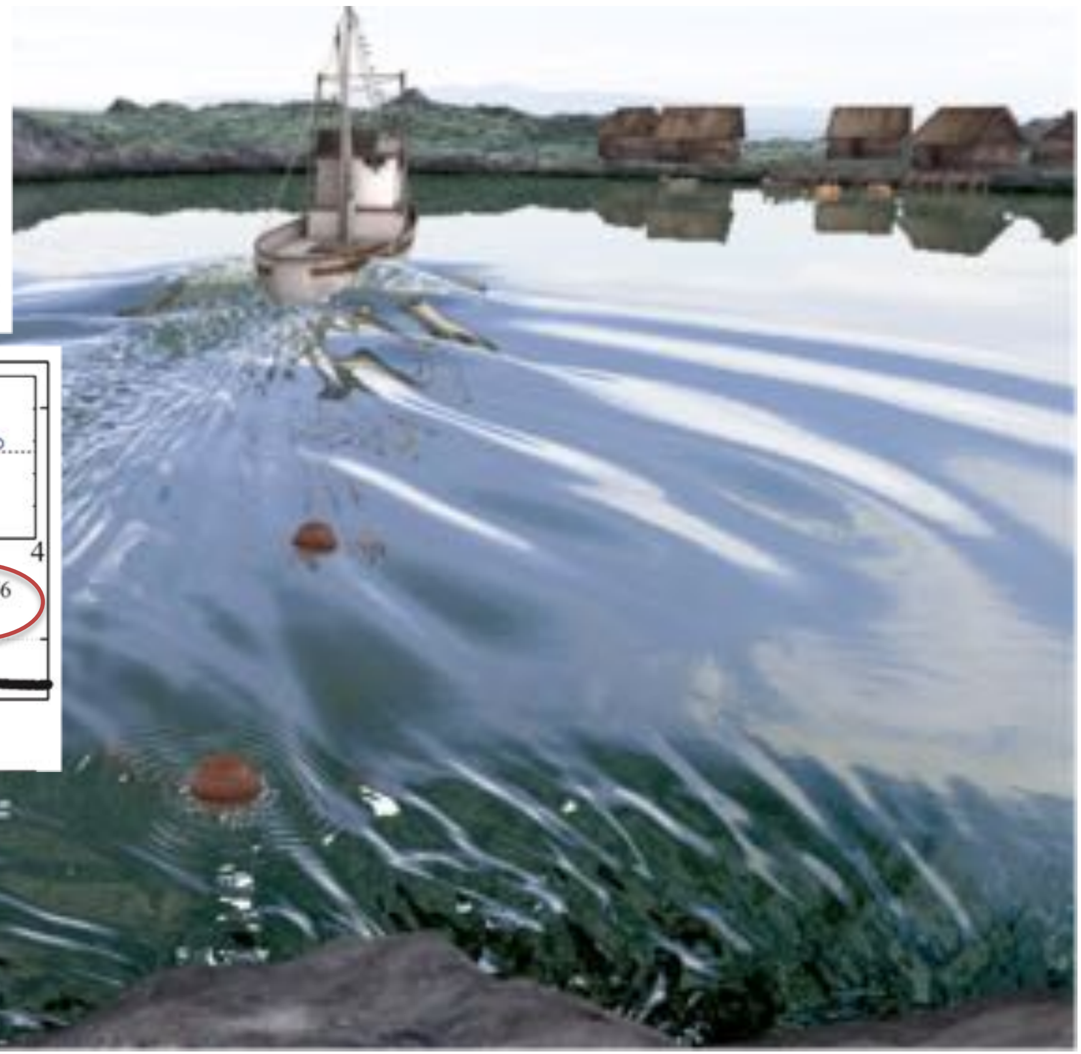
[Zakharov & Filonenko, JAMTP, 1967]

Capillary wave turbulence: experimental results

[e.g. Brazhnikov+, EPL, 2002; Falcon+, PRL, 2007; Boyer+, PRL, 2008; Cobelli+, PRL, 2011; Berhanu+, PRE, 2013; Aubourg & Mordant, PRL, 2015; Michel+, PRL, 2017; Berhanu+, JFM, 2018]







Back to cosmology

Weakly nonlinear general relativity

$\Lambda=0$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1 \qquad R_{\mu\nu} = 0$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \sout{R^{(3)}_{\mu\nu} + R^{(4)}_{\mu\nu} + \dots} \qquad R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$$

<u>Triadic interactions:</u>

$$\begin{cases} \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \text{ and } \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} \\ \omega_{\mathbf{k}} = c|\mathbf{k}| = ck \end{cases}$$

$\Rightarrow$ **Collinear wave vectors**

We found no contribution on the **resonant manifold**

Three-wave interactions in GW turbulence **do not contribute**!

[SG & Nazarenko, PRL, 2017]

Weakly nonlinear general relativity

$\Lambda=0$

Einstein-Hilbert action: $S = \frac{1}{2}\int R\sqrt{-g}\,d^4x$

g is the determinant of $g_{\mu\nu}$

R is the scalar curvature

Diagonal space-time metric:

$\partial/\partial z = 0$

$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

[Hadad & Zakharov, JGP, 2014]

$$H_0 = e^{-\lambda}\gamma,\ \ H_1 = e^{-\lambda}\beta,\ \ H_2 = e^{-\lambda}\alpha,\ \ H_3 = e^{\lambda}$$

(Lamé coefficients)

Lagrangian density:

Give the linear contribution

$$\Rightarrow \qquad \mathcal{L} = \frac{1}{2}\left[\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2 - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right]$$

$\alpha = \beta = \gamma = 1 \quad \lambda \ll 1 \quad \lambda = c_1\exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) + c_2\exp(i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x})$

Hadad & Zakharov's theorem (JGP, 2014)

- Dynamical equations given by:

$$\begin{cases} \dfrac{\delta S}{\delta \lambda} = 0 \\ \dfrac{\delta S}{\delta \alpha} = \dfrac{\delta S}{\delta \beta} = \dfrac{\delta S}{\delta \gamma} = 0 \end{cases}$$

4 equations

- Vacuum Einstein equations:

7 equations

$$[R_{\mu\nu}] = \begin{pmatrix} R_{00} & R_{01} & R_{02} & 0 \\ - & R_{11} & R_{12} & 0 \\ - & - & R_{22} & 0 \\ - & - & - & R_{33} \end{pmatrix} = [0]$$

It's compatible

Hamiltonian formalism

Normal variables: $\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \quad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}},$ (Fourier space)

Hamiltonian equation: $i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}}$ where $H = H_{\text{free}} + H_{\text{int}}$

$$H_{\text{free}} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2$$

With $R_{01}=R_{02}=R_{12}=0$ we find:

$$\begin{aligned}
H_{\text{int}} = \frac{1}{4} \sum_{1,2,3,4,5} \frac{\delta_{123}\delta^1_{45}}{\sqrt{k_2k_3k_4k_5}} \Bigg\{ \Bigg[\left(\frac{p_5}{p_1} + \frac{q_5}{q_1}\right) k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4}\right) + \frac{p_4q_5}{p_1q_1}(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] \\
k_2k_3(a_2 - a^*_{-2})(a_3 - a^*_{-3}) + \Bigg[-\left(\frac{p_5}{p_1} - \frac{q_5}{q_1}\right)(p_2p_3 - q_2q_3)k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4}\right) \\
+ \frac{p_4q_5}{p_1q_1}(\mathbf{k}_2 \cdot \mathbf{k}_3)(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] (a_2 + a^*_{-2})(a_3 + a^*_{-3}) \Bigg\} \\
+ \frac{1}{2} \sum_{\mathbf{k},1,2,3,4} \frac{\delta^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{34}}{\sqrt{k_1k_2k_3k_4}} \Bigg\{ \frac{(\mathbf{k} \cdot \mathbf{k}_2)k_1p_3q_4}{pq} \left(-\frac{a_1a_2 + a^*_{-1}a^*_{-2}}{k_2 + k_1} + \frac{a^*_{-1}a_2 + a_1a^*_{-2}}{k_2 - k_1}\right)(a^*_3 + a_{-3})(a^*_4 + a_{-4}) \\
+ \frac{k_1k_3p_2q_4}{pq}\left(a_1a_2 + a_1a^*_{-2} - a^*_{-1}a_2 - a^*_{-1}a^*_{-2}\right)\left(a^*_3a^*_4 + a^*_3a_{-4} - a_{-3}a^*_4 - a_{-3}a_{-4}\right) \Bigg\}.
\end{aligned}$$

4 wave processes

Kinetic equation of GW turbulence

$$n_{\mathbf{k}} = \langle |a_{\mathbf{k}}|^2 \rangle$$

$$H_{3\to 1} = 0$$ → Additional symmetry

$$\dot{n}_{\mathbf{k}} = 4\pi \int |T_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}\mathbf{k}_3}|^2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

$$\text{with } T_{34}^{12} = \tfrac{1}{4}(W_{34}^{12} + W_{34}^{21} + W_{43}^{12} + W_{43}^{21}),\ W_{34}^{12} = Q_{34}^{12} + Q_{12}^{34}$$

$$Q_{34}^{12} = \frac{1}{4\sqrt{k_1 k_2 k_3 k_4}} \left\{ 2\left(\frac{p_4}{p_1 - p_3} - \frac{q_4}{q_1 - q_3} \right) \frac{k_2(p_1 p_3 - q_1 q_3)}{k_1 - k_3} - 2\left(\frac{p_4}{p_1 - p_3} + \frac{q_4}{q_1 - q_3} \right) \frac{k_1 k_2 k_3}{k_1 - k_3} \right. \quad (12)$$
$$\left. + \left(\frac{p_2}{p_1 + p_2} - \frac{q_2}{q_1 + q_2} \right) \frac{k_1(p_3 p_4 - q_3 q_4)}{k_1 + k_2} - \left(\frac{p_2}{p_1 + p_2} + \frac{q_2}{q_1 + q_2} \right) \frac{k_1 k_3 k_4}{k_1 + k_2} + \frac{2 k_1 k_3 p_2 q_4}{(p_1 + p_2)(q_1 + q_2)} + \frac{2 k_1 p_3 (q_2 k_4 + k_2 q_4)}{(p_1 - p_3)(q_1 - q_3)} \right\}.$$

[SG & Nazarenko, PRL, 2017]

Constant flux (isotropic) spectra:

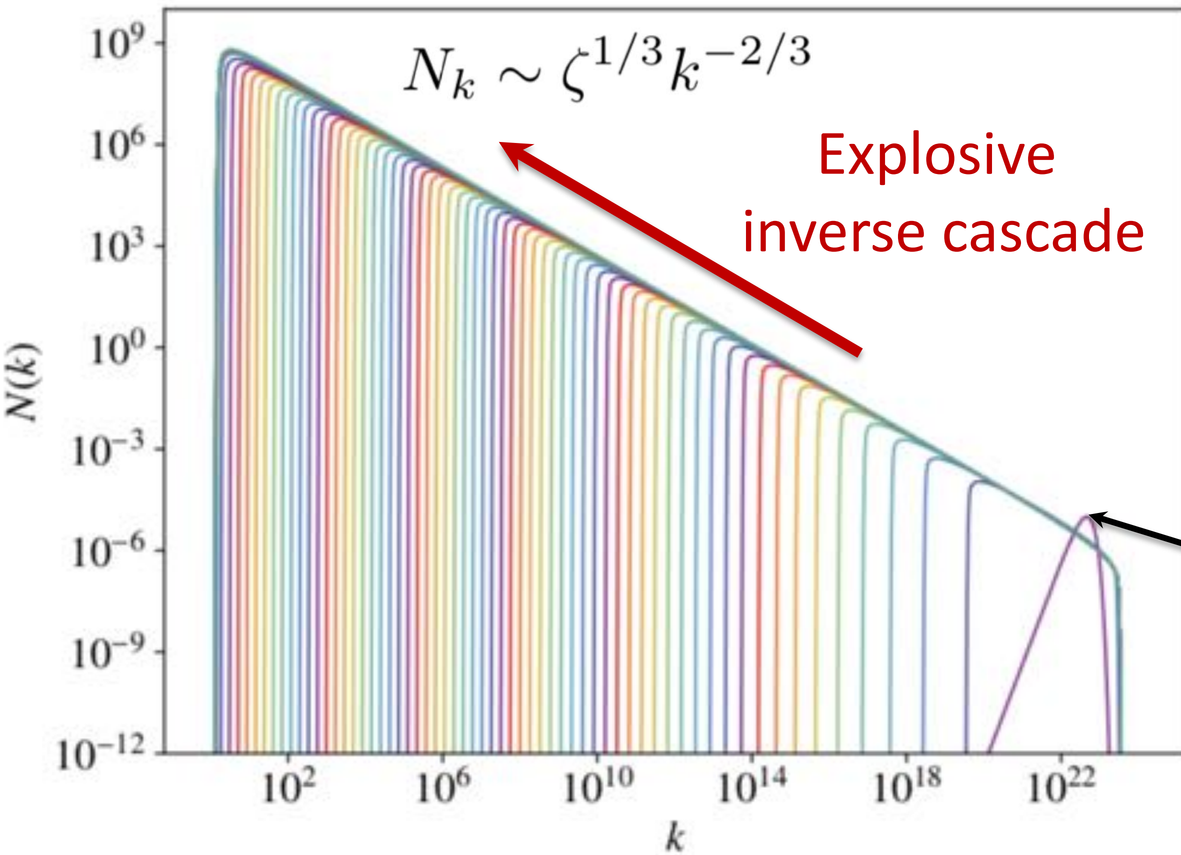
Energy	Wave action
$\mathcal{E} = \iint \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \mathrm{const}$	$\mathcal{N} = \iint n_{\mathbf{k}} d\mathbf{k} = \mathrm{const}$
$E_k^{(1D)} \sim \varepsilon^{1/3} k^0$	$N_k^{(1D)} \sim \zeta^{1/3} k^{-2/3}$ finite capacity
Direct cascade	**Inverse cascade**

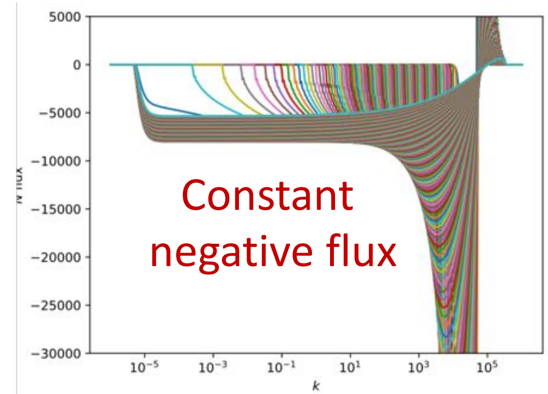
Local approximation: nonlinear diffusion model

- Rigorous derivation is rare (in MHD it's possible) [SG & Buchlin, ApJ, 2010]
- Here, it's a phenomenological model

[see also Dyachenko+, Physica D, 1992; Passot & Sulem, JPP, 2019]

$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k}\left[k^2 N^2(k) \frac{\partial (k N(k))}{\partial k}\right] - \nu k^4 N(k) - \eta \frac{N(k)}{k^4}$$

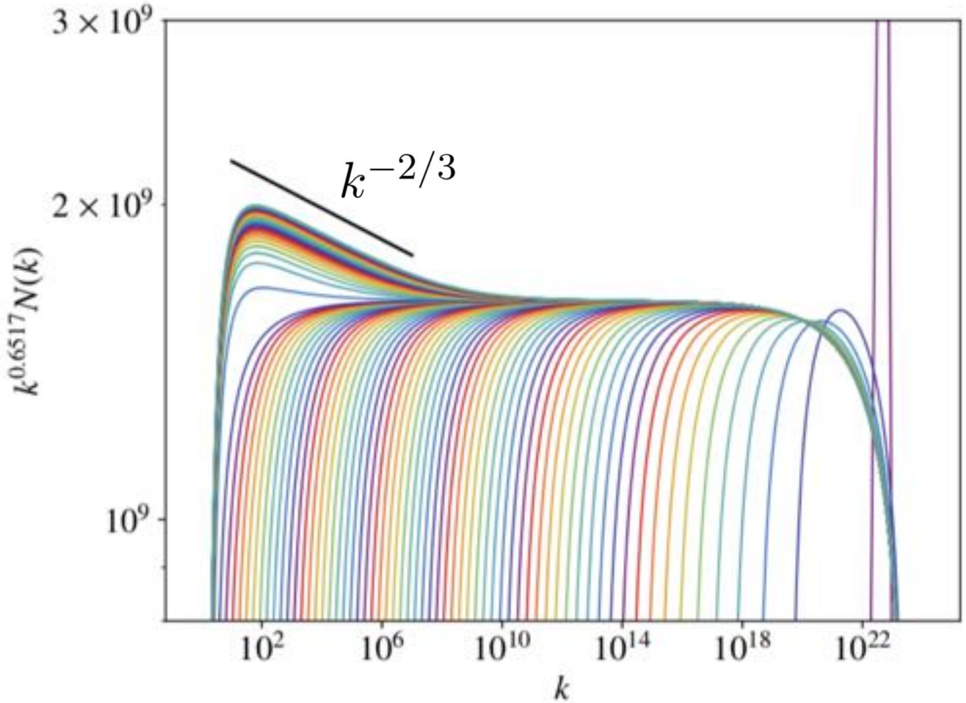




Initial spectrum (decay simulation)

[SG, Nazarenko, Buchlin & Thalabard, Physica D, 2019]

Anomalous scaling



Old subject (weak & strong)

[Semikoz & Tkachev, PRL, 1995]

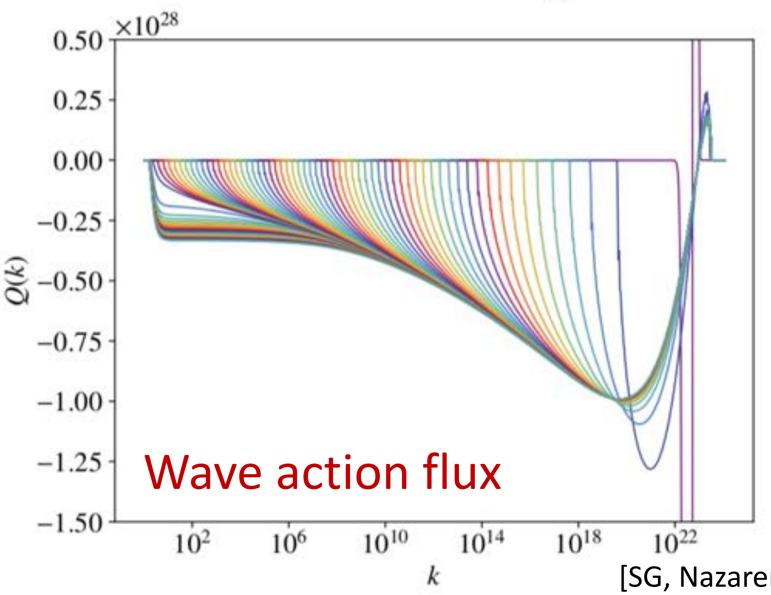
[SG+, JPP, 2000; Lacaze+, Physica D, 2001]

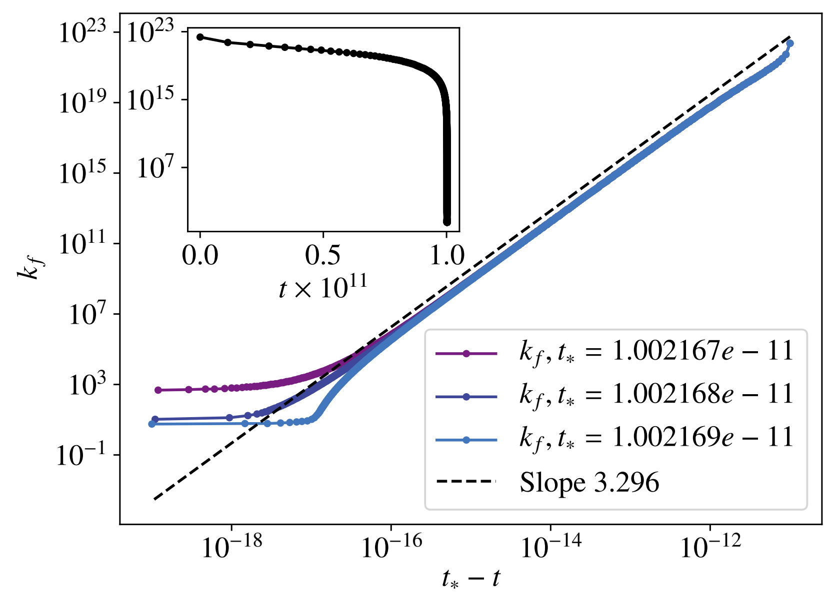
[see eg. Connaughton & Nazarenko, PRL, 2004; Nazarenko, JETPL, 2006; Boffetta+, JLTP, 2009]

[Thalabard+, JPAMT, 2015]

Self-similar solution of the second kind:

$$N = \frac{1}{\tau^{\alpha}} N_0\left(\frac{k}{\tau^{\beta}}\right) \qquad \tau = t_* - t$$





[SG, Nazarenko, Buchlin & Thalabard, Physica D, 2019]

$$k_f \sim (t_* - t)^{3.296}$$

Extrapolation/phenomenology

(beyond weak turbulence)

Big-Bang scenario driven by space-time turbulence ?

$t \approx 10^{-36}$ s

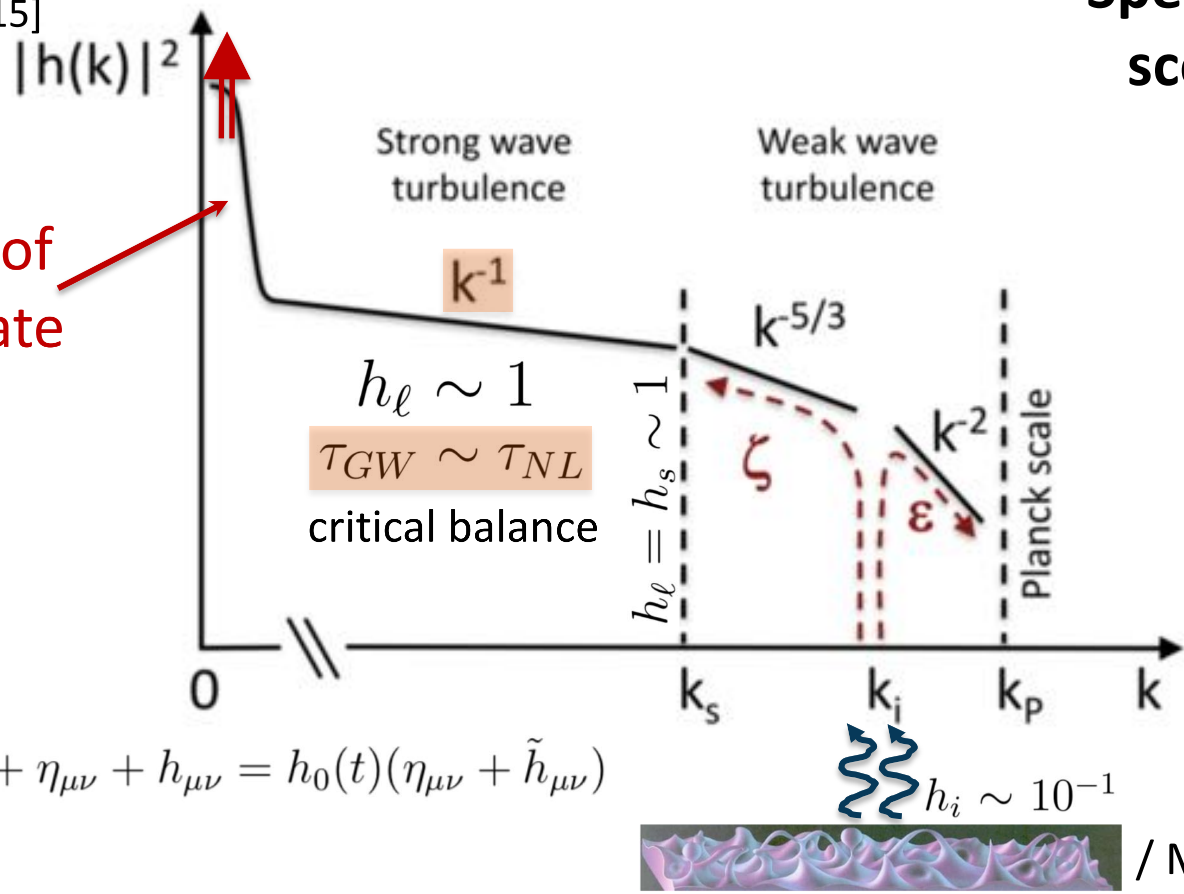
[SG, Laurie & Nazarenko, 2019]

[Nazarenko & Onorato, 2006; 2007]
[During+, PRE, 2015]

Speculative scenario

Formation of a condensate

$$\tau_{GW} \sim 1/\omega$$
$$\tau_{NL} \sim \ell/(h_\ell c)$$



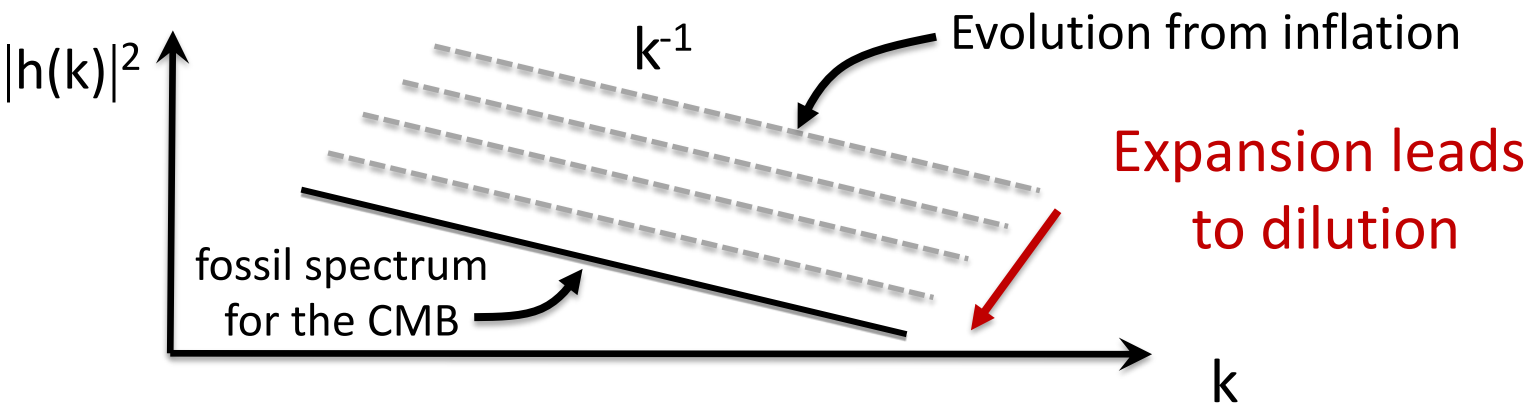
/ Merger of PBH

$$g_{\mu\nu} = h_0(t)\eta_{\mu\nu} + \eta_{\mu\nu} + h_{\mu\nu} = h_0(t)(\eta_{\mu\nu} + \tilde{h}_{\mu\nu})$$

The growth of $h_0(t)$ is interpreted as an expansion of the Universe

Inflation appears if the growth in time is fast enough

Comparison with observations



Small fluctuations are treated in the Newtonian limit: $\nabla^2\phi = 4\pi G\rho$

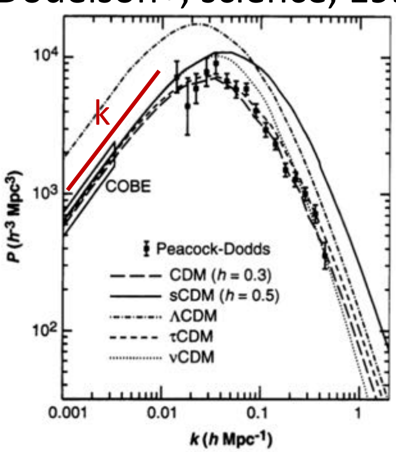
$$E_\ell \sim \frac{c^4}{32\pi G}\frac{h_\ell^2}{\ell^2} \Rightarrow P_\phi(k) = \phi^2(k) \sim k^{n_s-2} \sim k^{-1}$$

Prediction compatible with the Harrison-Zeldovich spectrum (n_s=1)

[Harrison, PRD, 1970; Zeldovich, MNRAS, 1972]

$P_\rho(\mathbf{k}) = \rho^2(\mathbf{k}) \sim \rho^2(k)/k^2 \sim k^{n_s} \sim k$

[Dodelson+, science, 1996]

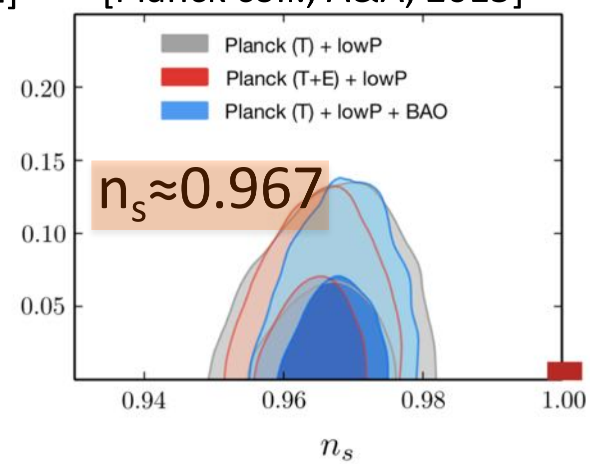


($\delta T/T \sim \delta\rho/\rho \sim 10^{-5}$)

Planck is compatible with the fossil spectrum $|h(k)|^2 \sim k^{-1.033}$

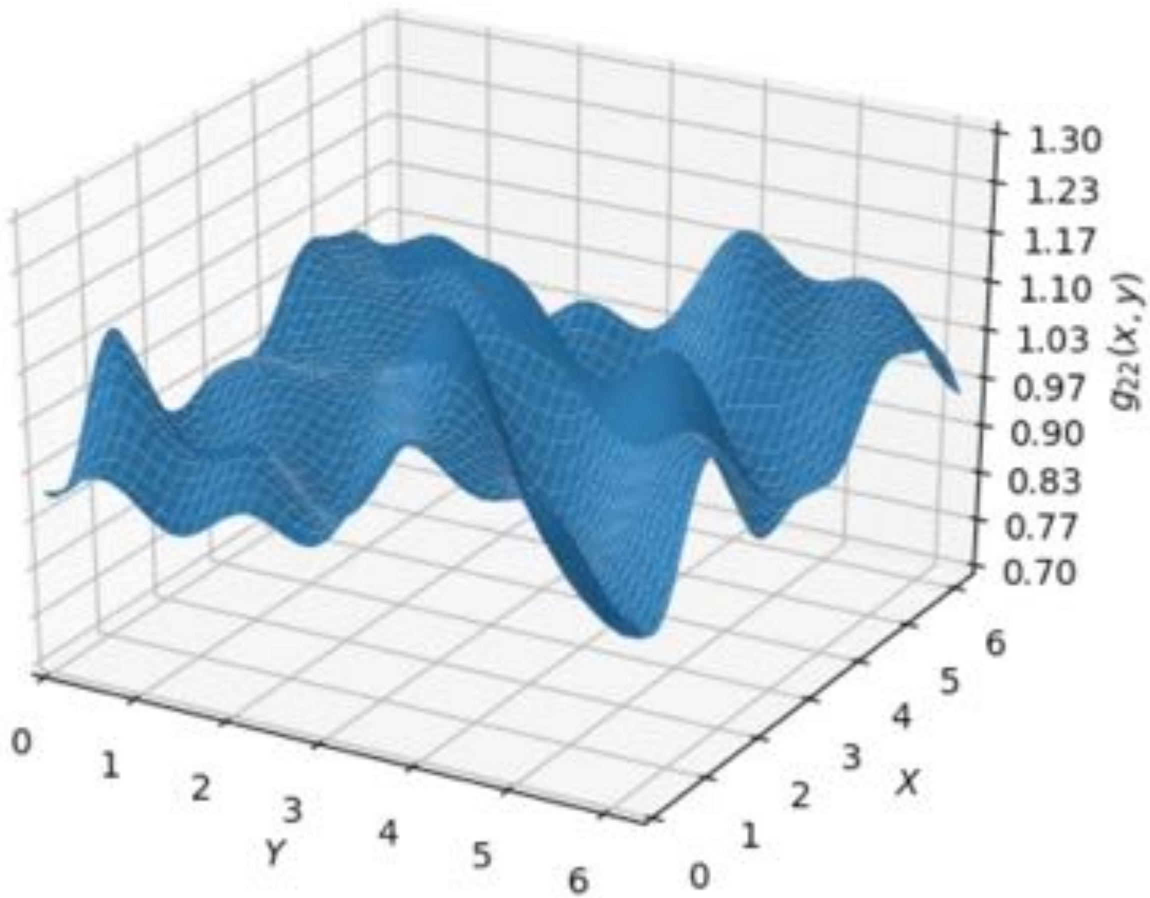
Latest data

[Planck coll., A&A, 2015]



Present/Future: direct numerical simulations

Pseudo-spectral code



Conclusion

- Theory of weak GW – space-time – turbulence (4 waves)
- Explosive inverse cascade of wave action / **anomalous** scaling
- The Riemann (4^{th} order) curvature tensor is **non-trivial**

- Phenomenological (CB) model of inflation (via a condensate)
  - ➔ a **standard** model of inflation ('no' tuning parameter)
  - ➔ **falsifiable** predictions (with dns of general relativity)
  - ➔ close to elastic wave turbulence [Hassaini+, 2018]
- **Fossil spectrum ~ compatible with CMB data**
- Presence of an **anomalous** scaling in the Planck data (n_s<1)
  - ➔ necessary to explain the structures in the Universe
  - ➔ can this anomalous scaling be explained by turbulence ?

[Semikoz & Tkachev, PRL, 1995 ; Lacaze+, PD, 2001]