# A unified asymptotic approach to fluid dynamical problems in rapidly rotating systems

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# Outline of the Talk

- Introduction: A unified asymptotic approach
- An example: Convection in rapidly rotating spheres
- An example: Precessing flow in rapidly rotating oblate spheroids
- Concluding remarks

## Introduction

**There are primarily three special characteristics in rapidly (the small Ekman number E) rotating fluids**:

- **An overwhelming constraint on fluid motions imposed by controlling rotational forces;**
- **A unique type of oscillatory motions, inertial oscillations and inertial waves, solely caused by the action of rotational forces;**
- **A viscous boundary layer, produced by the effect of fast rotation, that differs markedly from that in non-rotating configurations.**

# Introduction

• **In a unified asymptotic approach for a rapidly rotating system, the governing equations may be cast in the form**

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{u} + \frac{1}{\rho}\nabla p = \{\textbf{small and boundary terms...}\}\,, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\textbf{Other equations (such as Energy Equation)...} \tag{3}$$

**where the terms marked by brackets, which may contain the Poincaré force $\mathbf{r} \times (\partial \mathbf{\Omega}/\partial t)$ or the thermal buoyancy force $\mathbf{g}\alpha\Theta$ or the viscous force $\nu\nabla^2\mathbf{u}$,..., are either multiplied by small parameters or physically small because the Coriolis force dominates and controls the dynamics of a rapidly rotating fluid system.**

## Introduction

- A unified approach to the theory of rotating flows hinges on the existence of mathematically explicit/complete and relatively simple analytical solutions for the system in which the terms marked by brackets are neglected.

- The system supports waves/oscillations enabled solely by the Coriolis force in completely filled, rapidly rotating fluid systems.

## Introduction

**Significant progress can be made if we piece together the four essential ingredients that are not only unique but also most fundamental to fluid dynamics in rotating systems**:

**(i) inertial waves or inertial oscillations,**

**(ii) viscous boundary layers,**

**(iii) interplay between inertial waves/oscillations and viscous boundary layers, and**

**(iv) excitation/selection of inertial waves/oscillations by thermal instabilities or Poincaré or other forces**.

## An example of the unified asymptotic approach: Convection in rapidly rotating spheres with $0 < \mathbf{E} \ll 1$

- The most asymptotic and numerical studies point to a consensus that the preferred mode of convection in rapidly rotating spheres with $0 < \mathbf{E} \ll 1$ is axially nonsymmetric and equatorially symmetric (columnar-roll structure)(Busse 1970)

- This example presents an asymptotic theory describing axially symmetric and equatorially antisymmetric convection – whose flow velocity $\mathrm{u}$ and pressure $p$ are fully analytical, simple and in closed form – that is in quantitative agreement with the numerical solution.

## Mathematical Formulation

The problem of thermal convection in a sphere is governed by the dimensionless equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\hat{\mathbf{z}} \times \mathbf{u} + \nabla p = +R\Theta \mathbf{r} + E\nabla^2 \mathbf{u}, \tag{4}$$

$$(Pr/E)\left(\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta\right) = \mathbf{u} \cdot \mathbf{r} + \nabla^2 \Theta, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{6}$$

where $\Theta$ represents the deviation of the temperature from its static distribution, $p$ is the total pressure, and $\mathbf{u}$ is the velocity of convection. The equations are solved subject to perfectly conducting (isothermal), impenetrable and stress-free conditions

$$\frac{\partial(\hat{\boldsymbol{\phi}} \cdot \mathbf{u}/r)}{\partial r} = \frac{\partial(\hat{\boldsymbol{\theta}} \cdot \mathbf{u}/r)}{\partial r} = \hat{\mathbf{r}} \cdot \mathbf{u} = \Theta = 0 \quad \text{at} \quad r = 1. \tag{7}$$

## Asymptotic Analysis

**Torsional convection (where the PT Theorem is defeated by inertial effects) suggests the following expansion for $0 < \mathsf{E} \ll 1$:**

$$\mathbf{u} = [\mathbf{u}_0(r,\theta) + (\widetilde{\mathbf{u}} + \widehat{\mathbf{u}})]\, \mathrm{e}^{\mathrm{i}2\sigma t}, \tag{8}$$

$$p = [p_0(r,\theta) + (\widetilde{p} + \widehat{p})]\, \mathrm{e}^{\mathrm{i}2\sigma t}, \tag{9}$$

$$\Theta = \Theta_0(r,\theta)\mathrm{e}^{\mathrm{i}2\sigma t} + \dots, \tag{10}$$

$$\sigma = \sigma_0 + \sigma_1, \tag{11}$$

where $\sigma$ denotes the half frequency (the frequency $\omega = 2\sigma$) of torsional convection with $0 < |\sigma| < 1$, $\widehat{\mathbf{u}}$, $\widehat{p}$ and $\sigma_1$ represent small perturbations caused by viscous effects, to the leading-order solution $\mathbf{u}_0, p_0$ and $\sigma_0$, and satisfy $|\widehat{\mathbf{u}}| \ll |\mathbf{u}_0|$ and $|\sigma_1| \ll |\sigma_0|$, and $\widetilde{\mathbf{u}}$ denotes a weak viscous boundary flow on the bounding spherical surface.

# Asymptotic Analysis

**The leading-order problem describes thermal-inertial oscillation and is governed by the equations**

$$\mathrm{i}2\sigma_0\mathbf{u}_0(r,\theta) + 2\hat{\mathbf{z}} \times \mathbf{u}_0(r,\theta) + \nabla p_0(r,\theta) = 0, \tag{12}$$

$$\left(\nabla^2 - \frac{\mathrm{i}2\sigma_0 Pr}{E}\right)\Theta_0(r,\theta) + \mathbf{r}\cdot\mathbf{u}_0(r,\theta) = 0, \tag{13}$$

$$\nabla\cdot\mathbf{u}_0 = 0, \tag{14}$$

subject to the boundary conditions

$$\hat{\mathbf{r}}\cdot\mathbf{u}_0 = \Theta_0 = 0 \ \text{ at } r = 1.$$

**The exact analytical solution can be expressible as**

$$
\begin{aligned}
p_0 &= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\mathcal{C}_{kij}r^{2(i+j)+1}\sigma_0^{2i}(1-\sigma_0^2)^j\sin^{2j}\theta\cos^{2i+1}\theta,\\
\hat{\mathbf{r}}\cdot\mathbf{u}_0 &= -\frac{\mathrm{i}}{2}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\mathcal{C}_{kij}\left[\sigma_0^2(2i+2j+1)-(2i+1)\right]\\
&\times \left[r^{2(i+j)}\sigma_0^{2i-1}(1-\sigma_0^2)^{j-1}\sin^{2j}\theta\cos^{2i+1}\theta\right],\\
\hat{\boldsymbol{\theta}}\cdot\mathbf{u}_0 &= -\frac{\mathrm{i}}{2}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\mathcal{C}_{kij}\left[2j\sigma_0^2\cos^2\theta+(2i+1)(1-\sigma_0^2)\sin^2\theta\right]\\
&\times \left[r^{2(i+j)}\sigma_0^{2i-1}(1-\sigma_0^2)^{j-1}\sin^{2j-1}\theta\cos^{2i}\theta\right],\\
\hat{\boldsymbol{\phi}}\cdot\mathbf{u}_0 &= \frac{1}{2}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\mathcal{C}_{kij}r^{2(i+j)}\sigma_0^{2i}(1-\sigma_0^2)^{j-1}\,(2j)\sin^{2j-1}\theta\cos^{2i+1}\theta,
\end{aligned}
$$

## Asymptotic Analysis

**In the exact analytical solution, $k = 1, 2, 3, \ldots$ is regarded as a parameter of the solution and to be determined by the next-order problem**, $\mathcal{C}_{kij}$ is defined as

$$\mathcal{C}_{kij} = \frac{(-1)^{i+j}[2(k+i+j)+1]!!}{2^{j+1}(2i+1)!!(k-i-j)!i!(j!)^2},$$

and $\sigma_0$ is a solution of

$$\sum_{j=0}^{k} \left\{ \frac{(-1)^j [2(2k-j+1)]!}{j![2(k-j)]!(2k-j+1)!} \right\} \sigma_0^{2(k-j)} = 0, \qquad (15)$$

which has $2k$ real distinct solutions within $0 < |\sigma_0| < 1$ ( **$\sigma_0$ is also to be determined by the next-order problem)**.

## Asymptotic analysis

The next-order problem is described by

$$2\mathrm{i}\sigma_0\left(\widetilde{\mathbf{u}}+\widehat{\mathbf{u}}\right)+2\hat{\mathbf{z}}\times\left(\widetilde{\mathbf{u}}+\widehat{\mathbf{u}}\right)+\nabla\left(\widetilde{p}+\widehat{p}\right)=R\mathbf{r}\Theta_0+E\nabla^2\left[\mathbf{u}_0+\widetilde{\mathbf{u}}\right]-\mathrm{i}2\sigma_1\mathbf{u}_0,$$

$$\nabla\cdot\left(\widetilde{\mathbf{u}}+\widehat{\mathbf{u}}\right)=0,$$

subject to the stress-free boundary condition

$$\hat{\mathbf{r}}\cdot\left(\widetilde{\mathbf{u}}+\widehat{\mathbf{u}}\right)=0 \text{ and } \hat{\mathbf{r}}\times\nabla\times\left(\frac{\mathbf{u}_0+\widetilde{\mathbf{u}}}{r^2}\right)=\mathbf{0} \text{ at } r=1.$$

**The boundary-layer flow $\widetilde{\mathrm{u}}$, albeit weak, must be retained such that $(\mathrm{u}_0+\widetilde{\mathrm{u}})$ satisfies the stress-free boundary condition.**

## Asymptotic analysis

The solvability condition yields an expression for the Rayleigh number $R$:

$$R = R[\mathbf{u}_0(k), \sigma_0, E, Pr], \text{ a lengthy analytical expression.} \tag{16}$$

**The critical Rayleigh number R is determined by minimizing R over a manifold of $\mathrm{u}_0$ with different values of $k$ and $\sigma_0$ for a given value of E and Pr.**

# Asymptotic analysis: Results

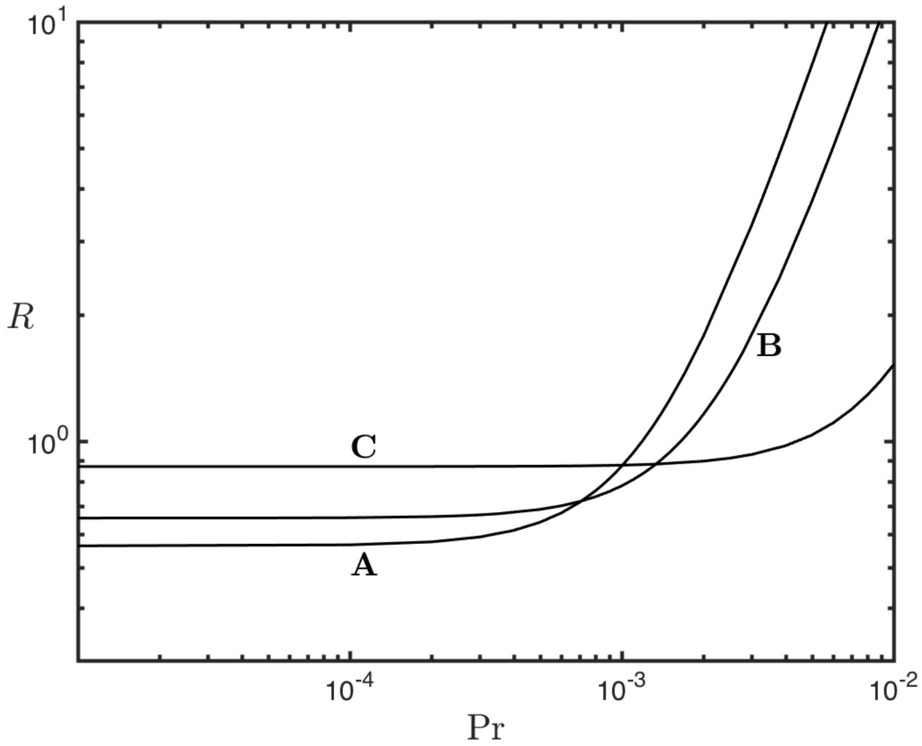


Figure 1: The curve $A$ represents solutions for retrogradely traveling waves with the azimuthal wavenumber $m = 1$; **the curve $B$ represents axially symmetric, equatorially antisymmetric, torsional convection** ; and the curve $C$ for progradely traveling waves with $m = 1$.

## Asymptotic analysis

The leading-order asymptotic solution for the pressure $p$ and the velocity $\mathbf{u}$ of convection is

$$
\begin{aligned}
p &= +\frac{3}{2}\left(1 - 2r^2 + \frac{5}{3}r^2\cos^2\theta\right) r\cos\theta \ \cos[(2/\sqrt{5})t], \\
\hat{\mathbf{r}}\cdot\mathbf{u} &= -\frac{3\sqrt{5}}{4}\left(1 - r^2\right)\cos\theta \ \sin[(2/\sqrt{5})t], \\
\hat{\boldsymbol{\theta}}\cdot\mathbf{u} &= -\frac{3\sqrt{5}}{4}\left(2r^2 - 1\right)\sin\theta \ \sin[(2/\sqrt{5})t], \\
\hat{\boldsymbol{\phi}}\cdot\mathbf{u} &= -\frac{15}{8}r^2\sin 2\theta \ \cos[(2/\sqrt{5})t],
\end{aligned}
$$

**It perhaps represents the simplest analytical solution of thermal convection in rapidly rotating fluid spheres with $0 < \mathbf{E} \ll 1$.**

## Asymptotic analysis

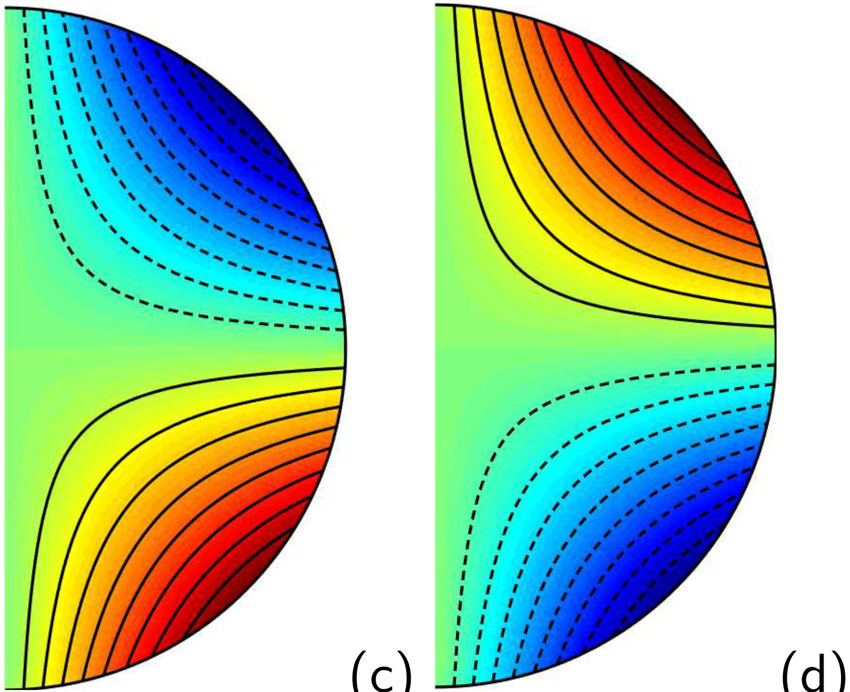


Figure 2: Contours of the axisymmetric azimuthal flow $\hat{\phi} \cdot \mathbf{u}$ in a meridional plane at two different oscillation states of torsional convection in rapidly rotating spheres. Solid contours indicate $\hat{\phi} \cdot \mathbf{u} > 0$ while dashed contours correspond to $\hat{\phi} \cdot \mathbf{u} < 0$.

## Nonlinear Properties

- When convective instability is characterized by axially nonsymmetric, equatorially symmetric and azimuthally traveling waves such as columnar rolls, the nonlinear convection near the threshold has the same equatorial symmetry with its properties being largely predictable based on the characteristics of the onset.

- When convective instability is marked by axially symmetric and equatorially antisymmetric oscillation, the equatorial symmetry of the nonlinear convection must be broken and it becomes difficult to predict its nonlinear properties based on the characteristics of the onset.

## Nonlinear properties: Three Different Solutions

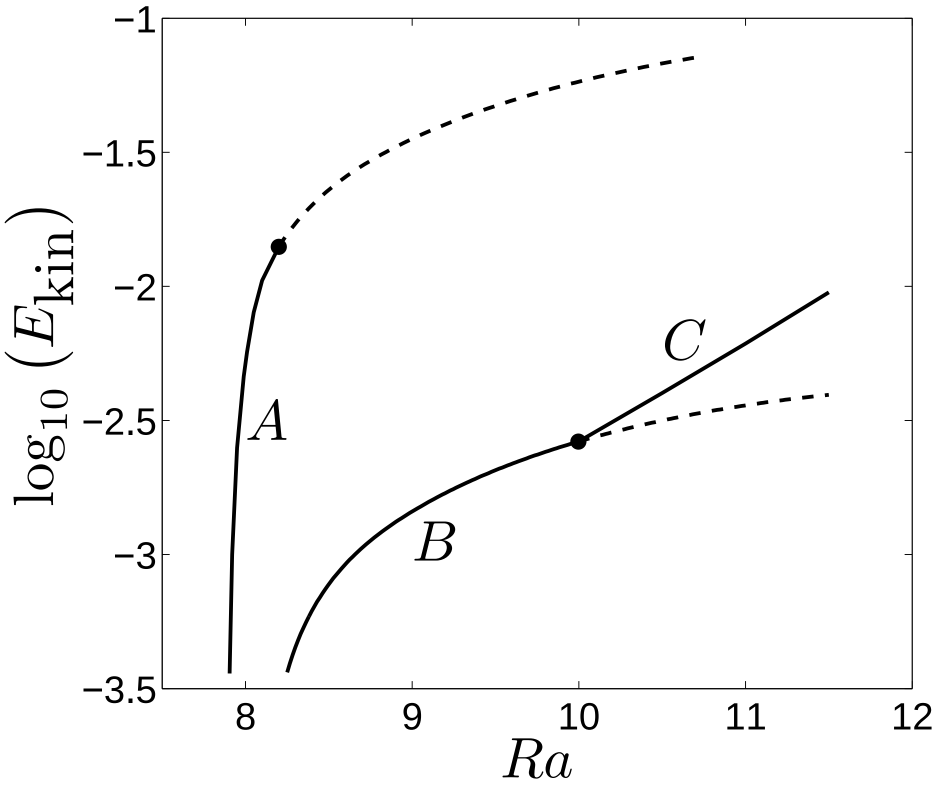


Figure 3: Branch A for axially axisymmetric torsional oscillatory waves; Branch B for axially non-axisymmetric and equatorially symmetric, thermal inertial waves; and Branch C for thermal inertial waves modified by the torsional oscillatory wave.

# Nonlinear properties: Primary Solution

**An interesting nonlinear phenomenon of thermal convection in rotating spheres:**

**Axially symmetric, equatorially non-symmetric, latitudinally propagating nonlinear waves.**

# Nonlinear properties: Primary Solution

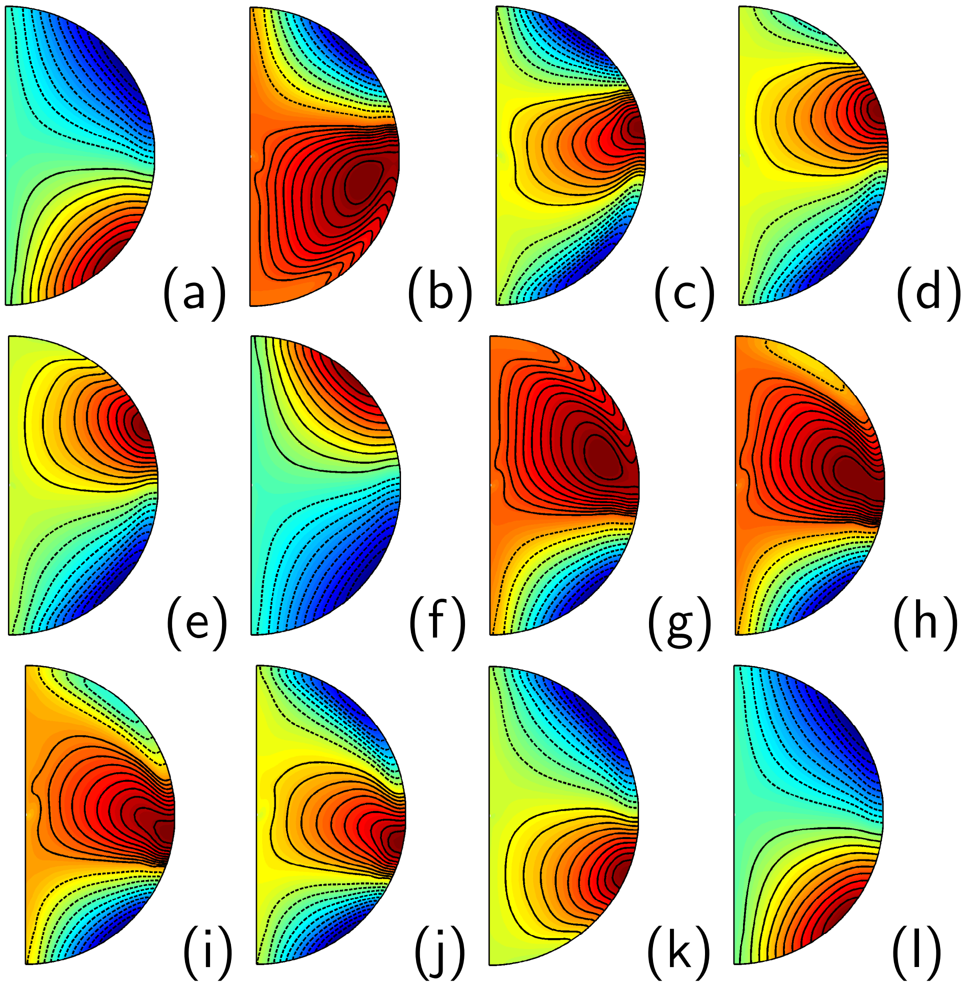


Figure 4: Contours of $\hat{\phi} \cdot \mathbf{u}$ in a meridional plane at 12 different instants.

# Nonlinear Properties: Secondary solution

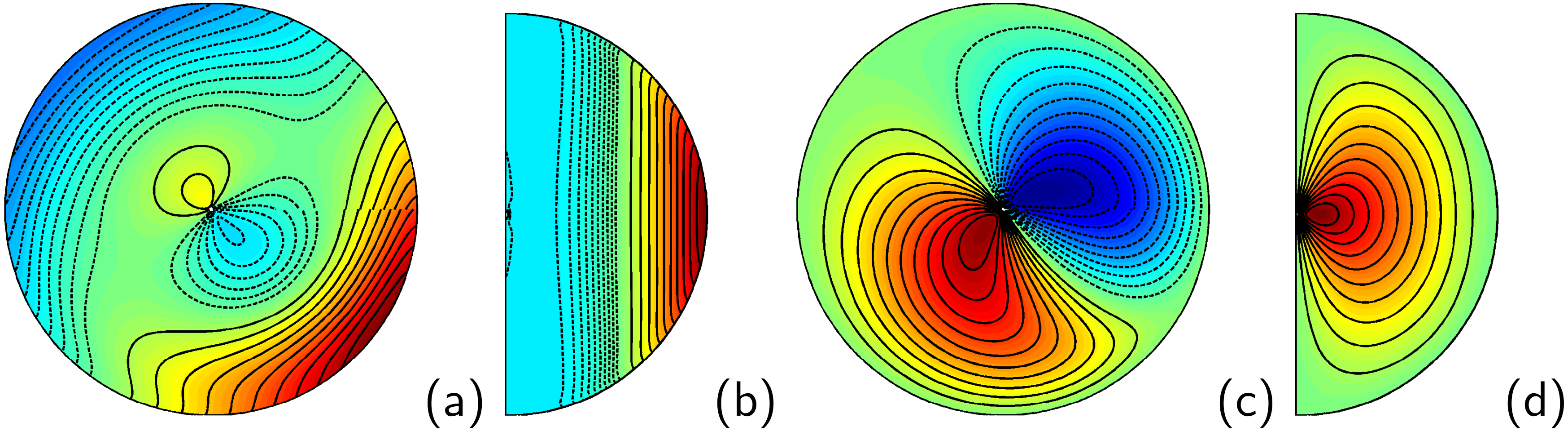


Figure 5: Contours of (a) $\hat{\boldsymbol{\phi}} \cdot \mathbf{u}_0$ and (c) $\hat{\mathbf{r}} \cdot \mathbf{u}$ in the equatorial plane and (b) $\hat{\boldsymbol{\phi}} \cdot \mathbf{u}$ and (d) $\hat{\mathbf{r}} \cdot \mathbf{u}$ in a meridional plane.

## Nonlinear Properties: Secondary solution

The convective flow is axially asymmetric, equatorially symmetric, propagates in the eastward (prograde) direction, and its leading-order solution is given by

$$
\begin{aligned}
p &= -\frac{8\mathcal{A}}{135}\left(\sqrt{10}+5\right) r\sin\theta \\
&\times \left[5\left(\sqrt{10}-1\right) r^2\cos(2\theta) + 3\left(\sqrt{10}-7\right) r^2 + 18\right] \mathrm{e}^{\mathrm{i}[\phi+(2/3)\left(1-2\sqrt{2/5}\right)t]}, \\
\hat{\mathbf{r}}\cdot\mathbf{u} &= -4\mathcal{A}\mathrm{i}\left(r^2-1\right)\sin\theta \mathrm{e}^{\mathrm{i}[\phi+(2/3)\left(1-2\sqrt{2/5}\right)t]}, \\
\hat{\boldsymbol{\theta}}\cdot\mathbf{u} &= \frac{4\mathcal{A}}{9}\mathrm{i}\left[\left(4\sqrt{10}-13\right) r^2 + 9\right]\cos\theta \mathrm{e}^{\mathrm{i}[\phi+(2/3)\left(1-2\sqrt{2/5}\right)t]}, \\
\hat{\boldsymbol{\phi}}\cdot\mathbf{u} &= -\frac{4\mathcal{A}}{9}\left[\left(4\sqrt{10}+5\right) r^2\cos(2\theta) - 18r^2 + 9\right] \mathrm{e}^{\mathrm{i}[\phi+(2/3)\left(1-2\sqrt{2/5}\right)t]}.
\end{aligned}
$$

# Nonlinear Properties: Tertiary solution

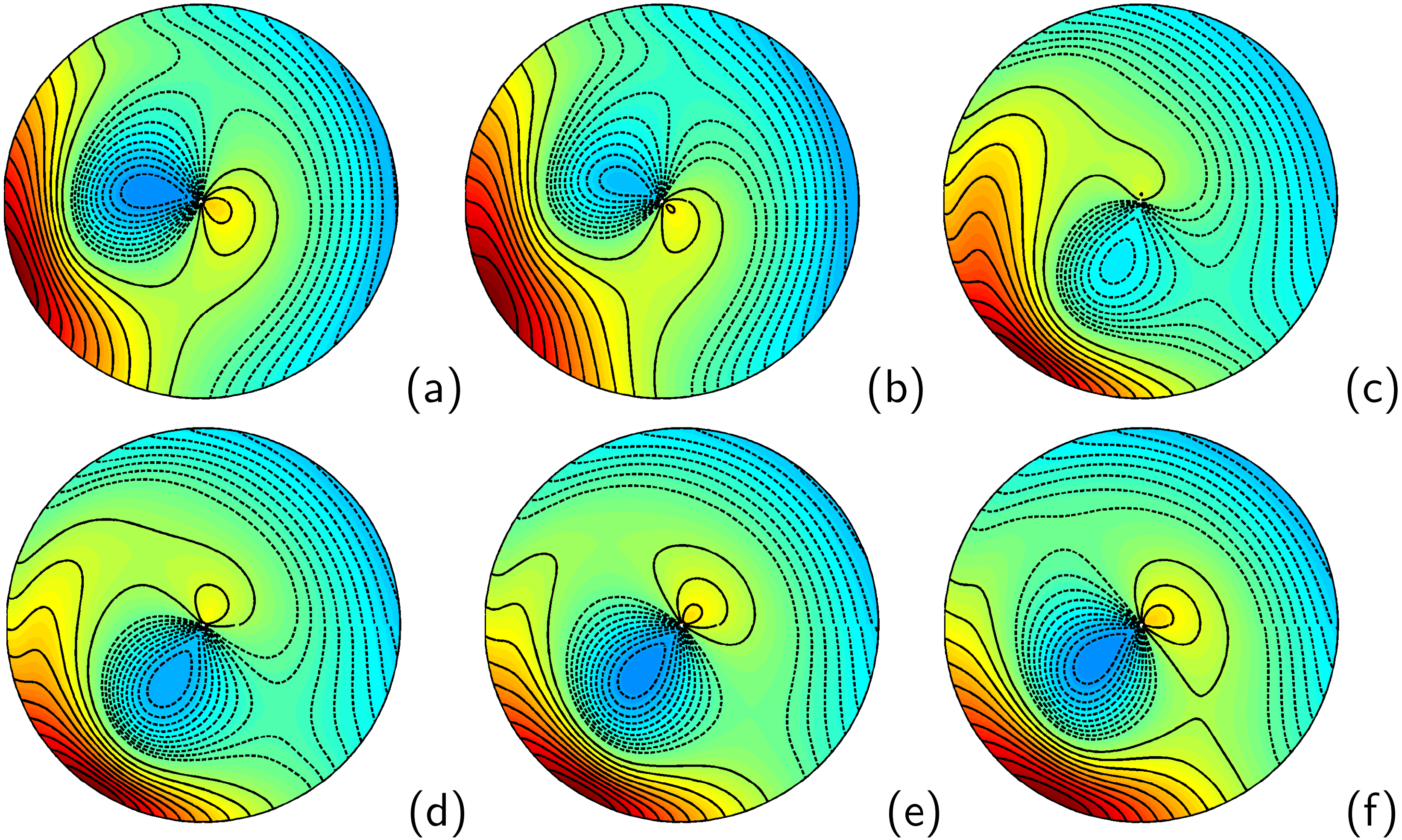


Figure 6: Contours of $\hat{\phi} \cdot \mathbf{u}$ in the equatorial plane.

# Nonlinear properties: Tertiary solution

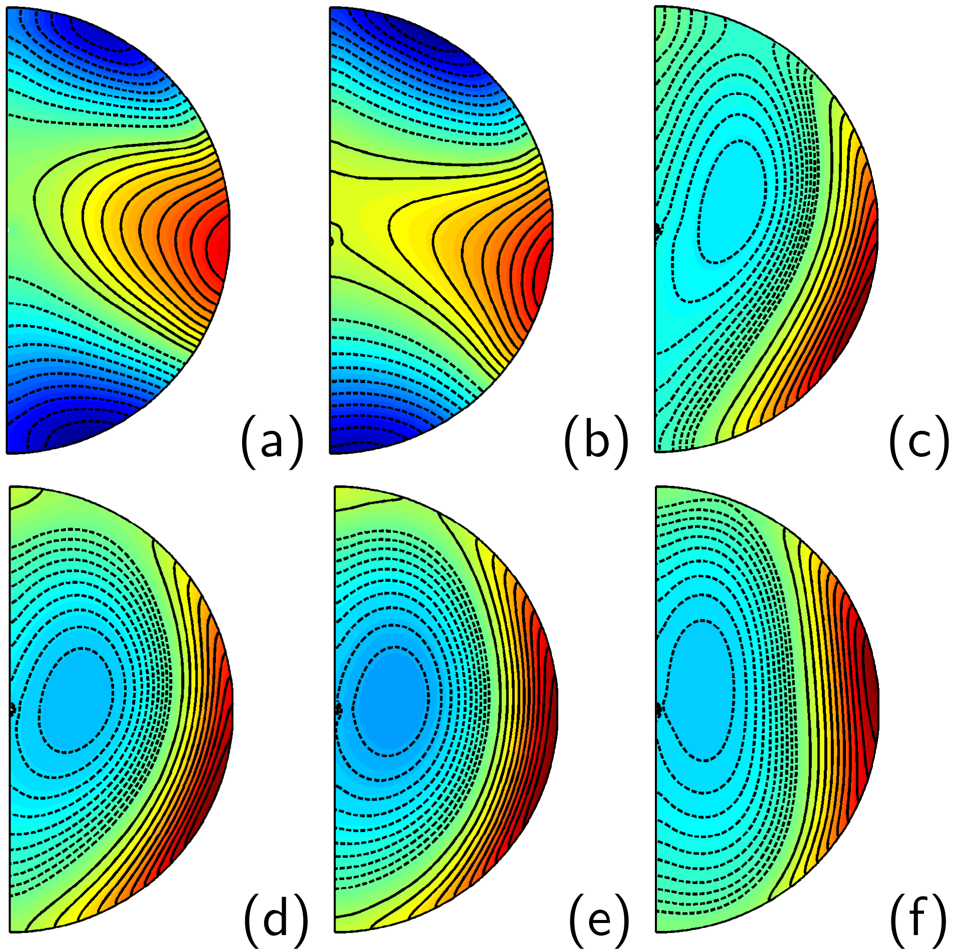

Figure 7: Contours of $\hat{\phi} \cdot \mathbf{u}$ in the meridional plane.

## An example of the unified asymptotic approach: Precessing flow in rapidly rotating spheroids

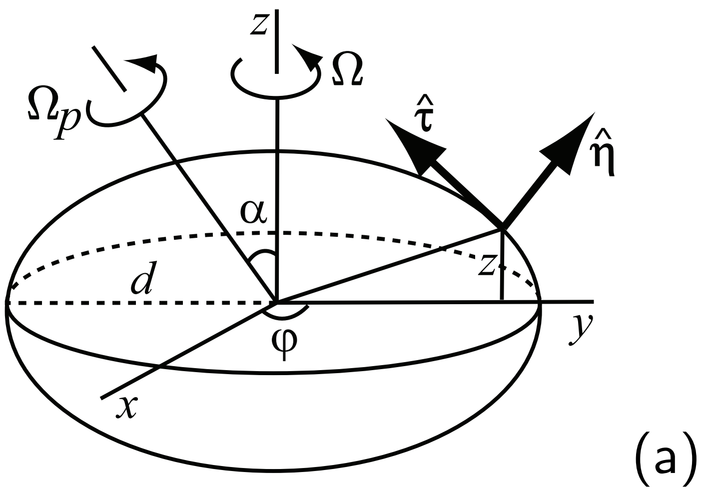


Figure 8: Geometry of a rotating fluid spheroidal cavity of semi-major axis $d$ and semi-minor axis $d\sqrt{1-\mathcal{E}^2}$ with spheroidal coordinates $(\eta, \phi, \tau)$ and corresponding unit vector $(\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\tau}})$.

## Mathematical Equations in the Mantle Frame

Consider a spheroidal cavity of arbitrary eccentricity rotating rapidly with angular velocity $\hat{\mathbf{z}}\Omega$ and, at the same time, precesses slowly with angular velocity $\boldsymbol{\Omega}_p$ that is fixed in space and inclined at an angle $\alpha$ $(0 < \alpha < \pi/2)$ with $\hat{\mathbf{z}}$. In the mantle frame of reference, processionally driven flows are governed by:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &+ \mathbf{u}\cdot\nabla\mathbf{u} + 2\left(\hat{\mathbf{z}} + Po\widehat{\boldsymbol{\Omega}}_p\right)\times\mathbf{u} + \nabla p \\
&= E\nabla^2\mathbf{u} + Po\mathbf{r}\times\left(\widehat{\boldsymbol{\Omega}}_p\times\hat{\mathbf{z}}\right), \qquad (17) \\
\nabla\cdot\mathbf{u} &= 0, \qquad (18)
\end{aligned}
$$

where the Ekman number, $E = \nu/\Omega r_o^2$, and the Poincaré number, $Po = \pm|\boldsymbol{\Omega}_p|/\Omega$

## Mathematical Equations in the Mantle Frame

In the mantle frame, the dimensionless precession vector $\widehat{\boldsymbol{\Omega}}_p$ is time-dependent and is given by

$$\widehat{\boldsymbol{\Omega}}_p = \sin\alpha\,(\hat{\mathbf{x}}\cos t - \hat{\mathbf{y}}\sin t) + \hat{\mathbf{z}}\cos\alpha, \tag{19}$$

The precessing flow on the bounding surface, $\mathcal{S}$, of the container is at rest, which requires

$$\hat{\mathbf{n}}\cdot\mathbf{u} = \hat{\mathbf{n}}\times\mathbf{u} = 0. \tag{20}$$

The problem will be solved asymptotically for an arbitrarily small but fixed $E$ with sufficiently small $Po$ and numerically for both weakly and strongly precessing flows.

## Mathematical Equations in the Mantle Frame

**We shall employ oblate spheroidal coordinates $(\eta, \phi, \tau)$ defined by**

$$
\begin{aligned}
x^2 &= (\mathcal{E}^2 + \eta^2)(1 - \tau^2)\cos^2\phi, \\
y^2 &= (\mathcal{E}^2 + \eta^2)(1 - \tau^2)\sin^2\phi, \\
z^2 &= \eta^2\tau^2,
\end{aligned}
$$

where $z$ is at the axis of $\hat{\mathbf{z}}\Omega$. The spheroidal polar coordinates $(\eta, \phi, \tau)$ are then restricted by

$$0 \le \eta \le \sqrt{1 - \mathcal{E}^2}, \;\; 0 \le \phi \le 2\pi, \;\; -1 \le \tau \le +1.$$

**The envelope of the spheroidal cavity is simply described by**

$$\eta = \sqrt{1 - \mathcal{E}^2}.$$

## Mathematical Equations in the Mantle Frame

**In our asymptotic analysis, we shall assume that precessing flows are weak with a small Poincaré number** $Po$:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{2(1+Po\cos\alpha)}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\left(\hat{\boldsymbol{\eta}}\tau\sqrt{\mathcal{E}^2+\eta^2}+\hat{\boldsymbol{\tau}}\eta\sqrt{1-\tau^2}\right)\times\mathbf{u}+\nabla P$$

$$= E\nabla^2\mathbf{u} - \frac{2Po\sin\alpha\sqrt{(\mathcal{E}^2+\eta^2)(1-\tau^2)}}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}$$

$$\times \left(\hat{\boldsymbol{\eta}}\tau\sqrt{\mathcal{E}^2+\eta^2}+\hat{\boldsymbol{\tau}}\eta\sqrt{1-\tau^2}\right)\cos(\phi+t), \quad (21)$$

$$\nabla\cdot\mathbf{u} = 0, \quad (22)$$

where $P$ denotes a reduced pressure including all the gradient terms. **As we shall demonstrate later, solutions of the above equations actually provides a satisfactory quantitative agreement with the fully nonlinear solutions including the full nonlinear simulation.**

## Asymptotic Analysis in the Mantle Frame

**The analysis is based on the following physical and mathematical observations:**

- **The analytical solution in the limit $Po = 0$ and $E = 0$ is available (Zhang et al., 2004, JFM), providing the necessary framework to construct the leading-order solution for the interior precessing flow in the mantle frame.**

- **Naturally, an asymptotic problem describing the precessing flows for $|Po \ll 1$ and $E \ll 1$ may be regarded as a small perturbation to the leading-order solution at $Po = 0$ and $E = 0$, forming a mathematically tractable asymptotic problem.**

- **If we treat the Poincaré forcing the same way we treat the buoyancy force in convection, the asymptotic method used in convection (Zhang et al. 2007, JFM) can be modified for solving the precession problem.**

## Asymptotic Analysis in the Mantle Frame

**Those observations lead to an asymptotic expansion in the mantle frame:**

$$\mathbf{u} \;=\; \sum_{mnk} \mathcal{A}_{mnk}\Big\{ \left(\mathbf{u}_{mnk} + \widehat{\mathbf{u}}_{mnk}\right) + \left[\widetilde{\mathbf{u}}_{mnk} + \left(\hat{\boldsymbol{\eta}} \cdot \widehat{\widetilde{\mathbf{u}}}_{mnk}\right)\hat{\boldsymbol{\eta}}\right]\Big\} \mathrm{e}^{\mathrm{i}t},$$

**where** $\mathrm{i} = \sqrt{-1}$**,** $\mathcal{A}_{mnk}$ **is the complex coefficients to be determined and a spheroidal inertial mode** $\mathbf{u}_{mkn}$ **satisfies**

$$\mathrm{i}2\sigma_{mnk}\mathbf{u}_{mnk} + 2\hat{\mathbf{z}} \times \mathbf{u}_{mnk} + \nabla p_{mnk} = 0,$$

$$\nabla \cdot \mathbf{u}_{mnk} = 0,$$

**where** $\sigma_{mnk}$ **is the half-frequency of the mode with** $|\sigma_{mnk}| < 1$**, subject to the inviscid boundary condition** $\hat{\eta} \cdot \mathbf{u}_{mnk} = 0$ **on** $\mathcal{S}$**.**

## Asymptotic Analysis in the Mantle Frame

**In the asymptotic expansion, viscous action on each inertial mode $\mathrm{u}_{mnk}$ induces a thin viscous boundary layer on $\mathcal{S}$ with its major tangential component $\widetilde{\mathrm{u}}_{mnk}$ (*i.e.*, $\hat{\mathrm{r}} \cdot \widetilde{\mathrm{u}}_{mnk} = 0$) and its small normal component $\hat{\mathrm{n}} \cdot \widehat{\widetilde{\mathrm{u}}}_{mnk}$.**

**By producing the normal mass flux from or sucking the interior fluid into the thin boundary layer, the viscous effect drives the secondary interior flow $\widehat{\mathrm{u}}_{mnk}$ and communicates to the interior fluid. It should be pointed out that the asymptotic expansion may be used to represent the spatial structure of any physically acceptable flow.**

**The analytical expressions for $\mathrm{u}_{mnk}$ and $p_{mnk}$ are given by**

# Asymptotic Analysis in the Mantle Frame

$$
\begin{aligned}
\hat{\boldsymbol{\eta}} \cdot \mathbf{u}_{mnk} &= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{\mathcal{C}_{mkij}}{2\sqrt{(\eta^2 + \mathcal{E}^2\tau^2)(\eta^2 + \mathcal{E}^2)}} \sigma^{2i-1}(1-\sigma^2)^{j-1} \\
&\times \left[-\eta^2\sigma(2j\sigma + m\sigma + m) + (2i+1)(\eta^2 + \mathcal{E}^2)(1-\sigma^2)\right] \\
&\times \left[(1-\tau^2)(\eta^2 + \mathcal{E}^2)\right]^{(m/2+j)} \eta^{2i}\tau^{2i+1}\mathsf{e}^{\mathsf{i}m\phi}, \\
\hat{\boldsymbol{\phi}} \cdot \mathbf{u}_{mnk} &= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{\mathcal{C}_{mkij}}{2\sqrt{(1-\tau^2)(\eta^2 + \mathcal{E}^2)}} \sigma^{2i}(1-\sigma^2)^{j-1}(2j + m + m\sigma) \\
&\times \left[(1-\tau^2)(\eta^2 + \mathcal{E}^2)\right]^{(m/2+j)} (\eta\tau)^{2i+1}\mathsf{e}^{\mathsf{i}m\phi}, \\
\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_{mnk} &= \mathsf{i}\sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{\mathcal{C}_{mkij}}{2\sqrt{(1-\tau^2)(\eta^2 + \mathcal{E}^2\tau^2)}} \sigma^{2i-1}(1-\sigma^2)^{j-1} \\
&\times \left[\tau^2\sigma(2j\sigma + m\sigma + m) + (2i+1)(1-\tau^2)(1-\sigma^2)\right] \\
&\times \left[(1-\tau^2)(\eta^2 + \mathcal{E}^2)\right]^{(m/2+j)} \eta^{2i+1}\tau^{2i}\mathsf{e}^{\mathsf{i}m\phi}.
\end{aligned}
$$

## Asymptotic Analysis in the Mantle Frame

**where $m \geq 0$, $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots, (2k+1)$, $\mathcal{C}_{mkij}$ is defined as**

$$\mathcal{C}_{mkij} = \left[\frac{-1}{(1-\sigma^2\mathcal{E}^2)}\right]^{i+j} \frac{[2(m+k+i+j)+1]!!}{2^{j+1}(2i+1)!!(k-i-j)!i!j!(m+j)!}.$$

Demanding that $\hat{\boldsymbol{\eta}} \cdot \mathbf{u} = 0$ at $\eta = \sqrt{1-\mathcal{E}^2}$ gives an equation for the half-frequency $\sigma$

$$\sum_{j=0}^{k} (-1)^j \frac{[2(2k+m-j+1)]!}{[2(k-j)+1]!j!(2k+m-j+1)!} [\ldots] \left[\frac{(1-\mathcal{E}^2)\sigma^2}{(1-\sigma^2\mathcal{E}^2)}\right]^{k-j} = 0.$$

**Here, for each given $k \geq 0$, the $2k+1$ distinct real solutions to can be arranged according to their size,**

$$0 < |\sigma_{mnk}| < |\sigma_{mnk}| < |\sigma_{mnk}|, \ldots, < |\sigma_{mnk}| < \ldots,$$

**with $\sigma_{mnk}$ for the $n$-th smallest solution and $1 \leq n \leq (2k+1)$.**

# Asymptotic Analysis in the Mantle Frame

Consider first the solution of the thin viscous boundary layer on $\mathcal{S}$, described by

$$\sum_{mnk} \mathcal{A}_{mnk}\Big[\mathrm{i}\widetilde{\mathbf{u}}_{mnk} + \frac{2}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\left(\hat{\boldsymbol{\eta}}\tau\sqrt{\mathcal{E}^2+\eta^2} + \hat{\boldsymbol{\tau}}\eta\sqrt{1-\tau^2}\right)\times\widetilde{\mathbf{u}}_{mnk}$$

$$+ \quad \hat{\boldsymbol{\eta}}\left(\frac{\partial\widetilde{p}_{mnk}}{\partial\xi}\right)E^{-1/2} - \frac{\partial^2\widetilde{\mathbf{u}}_{mnk}}{\partial\xi^2}\Big] = 0, \qquad (23)$$

$$\frac{\partial}{\partial\xi}\left(\sum_{mnk}\hat{\boldsymbol{\eta}}\cdot\widehat{\widetilde{\mathbf{u}}}_{mnk}\right) = E^{1/2}\hat{\boldsymbol{\eta}}\cdot\nabla\times\left[\hat{\boldsymbol{\eta}}\times\left(\sum_{mnk}\mathcal{A}_{mnk}\widetilde{\mathbf{u}}_{mnk}\right)\right]. \qquad (24)$$

Here $\xi$ is a stretched boundary-layer variable $\xi$ for which $\hat{\eta}\cdot\nabla = -E^{-1/2}\partial/\partial\xi$ and $\xi=\infty$ defines the outer edge of the thin boundary layer.

## Asymptotic Analysis in the Mantle Frame

**The boundary-layer equation is solved subject to the conditions:**

$$\left[\sum_{mnk} \mathcal{A}_{mnk} \widetilde{\mathbf{u}}_{mnk}\right]_{\xi=0} = -\left[\sum_{mnk} \mathcal{A}_{mnk} \mathbf{u}_{mnk}\right]_{\eta=\sqrt{1-\mathcal{E}^2}},$$

$$\left[\sum_{mnk} \mathcal{A}_{mnk} \frac{\partial^2(\hat{\boldsymbol{\tau}} \cdot \widetilde{\mathbf{u}}_{mnk})}{\partial \xi^2}\right]_{\xi=0} = -\left(1-\mathcal{E}^2+\mathcal{E}^2\tau^2\right)^{1/2}$$

$$\times \quad \sum_{mnk} \mathcal{A}_{mnk} \left[\mathrm{i}\left(1-\mathcal{E}^2+\mathcal{E}^2\tau^2\right)^{1/2} \hat{\boldsymbol{\tau}} \cdot \mathbf{u}_{mnk} + 2\tau \hat{\boldsymbol{\phi}} \cdot \mathbf{u}_{mnk}\right]_{\eta=\sqrt{1-\mathcal{E}^2}},$$

$$\left[\sum_{mnk} \mathcal{A}_{mnk} \frac{\partial^2(\hat{\boldsymbol{\phi}} \cdot \widetilde{\mathbf{u}}_{mnk})}{\partial \xi^2}\right]_{\xi=0} = -\left(1-\mathcal{E}^2+\mathcal{E}^2\tau^2\right)^{1/2}$$

$$\times \quad \sum_{mnk} \mathcal{A}_{mnk} \left[\mathrm{i}\left(1-\mathcal{E}^2+\mathcal{E}^2\tau^2\right)^{1/2} \hat{\boldsymbol{\phi}} \cdot \mathbf{u}_{mnk} - 2\tau \hat{\boldsymbol{\tau}} \cdot \mathbf{u}_{mnk}\right]_{\eta=\sqrt{1-\mathcal{E}^2}},$$

$$\left[\sum_{mnk} \mathcal{A}_{mnk} \widetilde{\mathbf{u}}_{mnk}\right]_{\xi=\infty} = \left[\sum_{mnk} \mathcal{A}_{mnk} \frac{\partial^2 \widetilde{\mathbf{u}}_{mnk}}{\partial \xi^2}\right]_{\eta=\sqrt{1-\mathcal{E}^2}} = 0.$$

## Asymptotic Analysis in the Mantle Frame

**The fourth-order equation can be readily solved but coefficients $\mathcal{A}_{mnk}$ remain undetermined.**

$$\hat{\boldsymbol{\tau}} \cdot \widetilde{\mathbf{u}} = \sum_{mnk} \frac{\mathcal{A}_{mnk}}{2} \left[ e^{\alpha^+ \xi} \left( \mathrm{i}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\tau}} \right) \cdot \mathbf{u}_{mnk} - e^{\alpha^- \xi} \left( \mathrm{i}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\tau}} \right) \cdot \mathbf{u}_{mnk} \right]_{\eta=\sqrt{1-\mathcal{E}^2}}$$

$$\hat{\boldsymbol{\phi}} \cdot \widetilde{\mathbf{u}} = \sum_{mnk} \frac{\mathcal{A}_{mnk}}{2} \left[ e^{\alpha^- \xi} \left( \mathrm{i}\hat{\boldsymbol{\tau}} - \hat{\boldsymbol{\phi}} \right) \cdot \mathbf{u}_{mnk} - e^{\alpha^+ \xi} \left( \hat{\boldsymbol{\phi}} + \mathrm{i}\hat{\boldsymbol{\tau}} \right) \cdot \mathbf{u}_{mnk} \right]_{\eta=\sqrt{1-\mathcal{E}^2}},$$

**where $\alpha^+$ and $\alpha^-$ are very complicated functions of $\tau$ and the coefficients $\mathcal{A}_{mnk}$ remain undetermined.**

## Asymptotic Analysis in the Mantle Frame

**The solvability condition for the interior problem:**

$$
\begin{aligned}
& \mathrm{i}\,(1-2\sigma_{mnk})\,\mathcal{A}_{mnk}\int_V |\mathbf{u}_{mnk}|^2\,\mathrm{d}V \\
+ \quad & E^{1/2}\sum_{m'n'k'}\mathcal{A}_{m'n'k'}\left\{\int_S [p^*_{mnk}]_{\eta=\sqrt{1-\mathcal{E}^2}}\,\hat{\boldsymbol{\eta}}\cdot\widetilde{\mathbf{u}}\right\}\mathrm{d}S \\
= \quad & Po\int_V \mathbf{u}^*_{mnk}\cdot\left[\mathbf{r}\times\left(\widehat{\boldsymbol{\Omega}}_p\times\hat{\mathbf{z}}\right)\right]\mathrm{d}V \\
- \quad & 2Po\cos\alpha\sum_{m'n'k'}\mathcal{A}_{m'n'k'}\int_V \mathbf{u}^*_{mnk}\times(\hat{\mathbf{z}}\times\mathbf{u}_{m'n'k'})\,\mathrm{d}V,
\end{aligned}
$$

where $\int_S$ denotes the surface integral over the envelope of a spheroidal cavity, $\int_V$ for the volume integral over the spheroid.

## Asymptotic Analysis in the Mantle Frame

**Here $p^*_{mnk}$ denotes the complex conjugate of $p_{mnk}$ and the indices, $m, n$ and $k$, take all permissable values. By performing direct integration, we find that**

$$\int_V \mathbf{u}^*_{mnk} \cdot \left[\mathbf{r} \times \left(\widehat{\mathbf{\Omega}}_p \times \hat{\mathbf{z}}\right)\right] \, \mathrm{d}V = (\mathrm{i}4\pi \sin\alpha)\, \mathcal{P}_k = (\mathrm{i}4\pi \sin\alpha)$$

$$\times \quad \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{(-1)^{i+j} \sigma_{1nk}^{2i-1} (1-\sigma_{1nk}^2)^j [2(i+j+k)+3]!!(1-\mathcal{E}^2)^{i+1/2}}{(k-i-j)!i!j!(2i+2j+5)!!(1-\mathcal{E}^2\sigma_{1nk}^2)^{i+j}}.$$

With several page of the analysis, we can prove that $\mathcal{P}_k = 0$ for all $k \geq 1$.

**It implies that the solvability system reduces to**

$$\mathcal{A}_{mnk} = 0, \quad \text{except for} \quad m = 1, \; n = 1, \; k = 0.$$

## Asymptotic Analysis in the Mantle Frame

**The non-zero coefficient $\mathcal{A}_{110}$ is then given by**

$$-\left(\frac{\mathrm{i}\mathcal{E}^2}{2-\mathcal{E}^2}\right)\mathcal{A}_{110}\int_V |\mathbf{u}_{110}|^2\,\mathrm{d}V + \mathcal{A}_{110}\left\{\int_S [p^*_{110}]_{\eta=\sqrt{1-\mathcal{E}^2}}\,\hat{\boldsymbol{\eta}}\cdot\widehat{\widehat{\mathbf{u}}}_{110}\,\mathrm{d}S\right\}$$

$$= \quad Po\frac{\mathrm{i}4\pi\sin\alpha}{5}(1-\mathcal{E}^2)^{1/2}(2-\mathcal{E}^2) + \left(\frac{\mathrm{i}2Po\cos\alpha\mathcal{A}_{110}}{2-\mathcal{E}^2}\right)\int_V |\mathbf{u}_{110}|^2\,\mathrm{d}V, \quad (25)$$

**which can be solved to determine $\mathcal{A}_{110}$ via some lengthy integrations if $\hat{\eta}\cdot\widehat{\widehat{\mathrm{u}}}_{110}$ is known.**

## Asymptotic Analysis in the Mantle Frame

**It can be shown that the mass flux takes the form**

$$\hat{\boldsymbol{\eta}} \cdot \widehat{\widetilde{\mathbf{u}}}_{110} = \frac{\mathcal{A}_{110} e^{\mathrm{i}\phi}}{(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}} \left\{ \frac{\partial}{\partial \tau} \left[ (1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2} (1-\tau^2)^{1/2} \mathcal{S}_1(\tau) \right] + \frac{\mathrm{i}(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)\mathcal{S}_2(\tau)}{(1-\tau^2)^{1/2}} \right\}$$

where

$$\mathcal{S}_1(\tau) = \frac{3\sqrt{2}(2-\mathcal{E}^2)}{16(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/4}\sqrt{1-\mathcal{E}^2}} \left\{ \frac{\mathrm{i}[\tau-(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}}{|(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}+2\tau|^{1/2}} + \frac{[\tau-(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}[(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}+2\tau]}{|(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}+2\tau|^{3/2}} + \frac{\mathrm{i}[\tau+(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}}{|(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}-2\tau|^{1/2}} + \frac{[\tau+(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}[(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}-2\tau]}{|(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/2}-2\tau|^{3/2}} \right., \ldots$$

## Asymptotic Analysis in the Mantle Frame: Solution

$$\begin{aligned}
\mathbf{u} \;&=\; \frac{\left[\mathrm{i}(15\sqrt{2}/4)E^{1/2}\mathcal{I}_r - \mathcal{E}^2(2-\mathcal{E}^2) - (15\sqrt{2}/4)E^{1/2}\mathcal{I}_i - 2Po(2-\mathcal{E}^2)\cos\alpha\right]}{\left[\mathcal{E}^2(2-\mathcal{E}^2) + (15\sqrt{2}/4)E^{1/2}\mathcal{I}_i + 2Po(2-\mathcal{E}^2)\cos\alpha\right]^2 + E(15\sqrt{2}\mathcal{I}_r^2/4)^2} \\
&\times\; \left[Po(2-\mathcal{E}^2)\sin\alpha\right] \left\{ \left[\frac{2\eta\sqrt{\eta^2+\mathcal{E}^2}(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\right]\hat{\boldsymbol{\tau}} \right. \\
&+\; \mathrm{i}(1-\mathcal{E}^2)^{1/2}\Big[(\tau - \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^+ E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)} \\
&-\; (\tau + \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^- E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\Big]\hat{\boldsymbol{\tau}} \\
&-\; 2\eta\tau\hat{\boldsymbol{\phi}} - \mathrm{i}(1-\mathcal{E}^2)^{1/2}\Big[(\tau - \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^+ E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)} \\
&+\; (\tau + \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^- E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\Big]\hat{\boldsymbol{\phi}} \\
&+\; \left.\left[\frac{2\mathrm{i}\tau\mathcal{E}^2\sqrt{1-\tau^2}(1-\mathcal{E}^2-\eta^2)}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\right]\hat{\boldsymbol{\eta}}\right\}\mathrm{e}^{\mathrm{i}(\phi+t)}. \qquad (26)
\end{aligned}$$

## Asymptotic Analysis in the Mantle Frame

**In the expression,**

$$\begin{aligned}
\mathcal{I}_r &= -\int_{-1}^{+1} (\tau - \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})(2\tau + \sqrt{1-\mathcal{E}^2+\mathcal{E}^2\tau^2}) \\
&\times \frac{(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/4}[(1-2\tau^2)+\tau\sqrt{1-\mathcal{E}^2+\mathcal{E}^2\tau^2})}{|2\tau+\sqrt{\eta^2+\mathcal{E}^2}|^{3/2}}\,\mathrm{d}\tau \\
\mathcal{I}_i &= -\int_{-1}^{+1} (\tau - \sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)^{1/4} \\
&\times \frac{(1-2\tau^2)+\tau\sqrt{1-\mathcal{E}^2+\mathcal{E}^2\tau^2})}{|2\tau+\sqrt{\eta^2+\mathcal{E}^2}|^{1/2}}\,\mathrm{d}\tau,
\end{aligned}$$

**For measuring the amplitude of the precessing flow, we introduce the kinetic energies of the precessing flows per volume**

$$E_{\text{kin}} = \frac{1}{2}\left(\frac{1}{V}\right)\int_V |\mathbf{u}|^2\,\mathrm{d}V.$$

## Asymptotic Explicit Solution in the Mantle Frame:

$$
\begin{aligned}
\mathbf{u} &= \frac{\left[\mathrm{i}4E^{1/2}g_1(\mathcal{E}) - \mathcal{E}^2(2-\mathcal{E}^2) - 4E^{1/2}g_2(\mathcal{E}) - 2Po(2-\mathcal{E}^2)\cos\alpha\right]}{\left[\mathcal{E}^2(2-\mathcal{E}^2) + 4E^{1/2}g_2(\mathcal{E}) + 2Po(2-\mathcal{E}^2)\cos\alpha\right]^2 + 16Eg_1^2(\mathcal{E})} \\
&\times \left[Po(2-\mathcal{E}^2)\sin\alpha\right]\left\{\left[\frac{2\eta\sqrt{\eta^2+\mathcal{E}^2}(1-\mathcal{E}^2+\tau^2\mathcal{E}^2)}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\right]\hat{\boldsymbol{\tau}}\right. \\
&+ \mathrm{i}(1-\mathcal{E}^2)^{1/2}\left[(\tau-\sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^+E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\right. \\
&- \left.(\tau+\sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^-E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\right]\hat{\boldsymbol{\tau}} \\
&- 2\eta\tau\hat{\boldsymbol{\phi}} - \mathrm{i}(1-\mathcal{E}^2)^{1/2}\left[(\tau-\sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^+E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\right. \\
&+ \left.(\tau+\sqrt{1-\mathcal{E}^2+\tau^2\mathcal{E}^2})\mathrm{e}^{\alpha^-E^{-1/2}(\sqrt{1-\mathcal{E}^2}-\eta)}\right]\hat{\boldsymbol{\phi}} \\
&+ \left.\left[\frac{2\mathrm{i}\tau\mathcal{E}^2\sqrt{1-\tau^2}(1-\mathcal{E}^2-\eta^2)}{\sqrt{\eta^2+\mathcal{E}^2\tau^2}}\right]\hat{\boldsymbol{\eta}}\right\}\mathrm{e}^{\mathrm{i}(\phi+t)},
\end{aligned}
$$

## Asymptotic Explicit Solution in the Mantle Frame

**In the analytical expression,**

$$
\begin{aligned}
g_1(\mathcal{E}) &= \frac{3}{56}\left(19\sqrt{2}+9\sqrt{6}\right)-\frac{\mathcal{E}^2}{308}\left(229\sqrt{2}+9\sqrt{6}\right)+O(\mathcal{E}^4)\\
g_2(\mathcal{E}) &= \frac{3}{56}\left(-19\sqrt{2}+9\sqrt{6}\right)-\frac{\mathcal{E}^2}{308}\left(229\sqrt{2}-9\sqrt{6}\right)+O(\mathcal{E}^4).
\end{aligned}
$$

The explicit formula for the kinetic energy of the precessing flow is

$$
E_{\rm kin} = \frac{(2/5)(Po\sin\alpha)^2(2-\mathcal{E}^2)^3(1-\mathcal{E}^2)}{\left[\mathcal{E}^2(2-\mathcal{E}^2)+4E^{1/2}g_2(\mathcal{E})+2Po(2-\mathcal{E}^2)\cos\alpha\right]^2+16Eg_1^2(\mathcal{E})}. \quad (27)
$$

**This explicit analytical formulas can be readily/directly used for planets.**

## Asymptotics vs. Numerics in the Mantle Frame

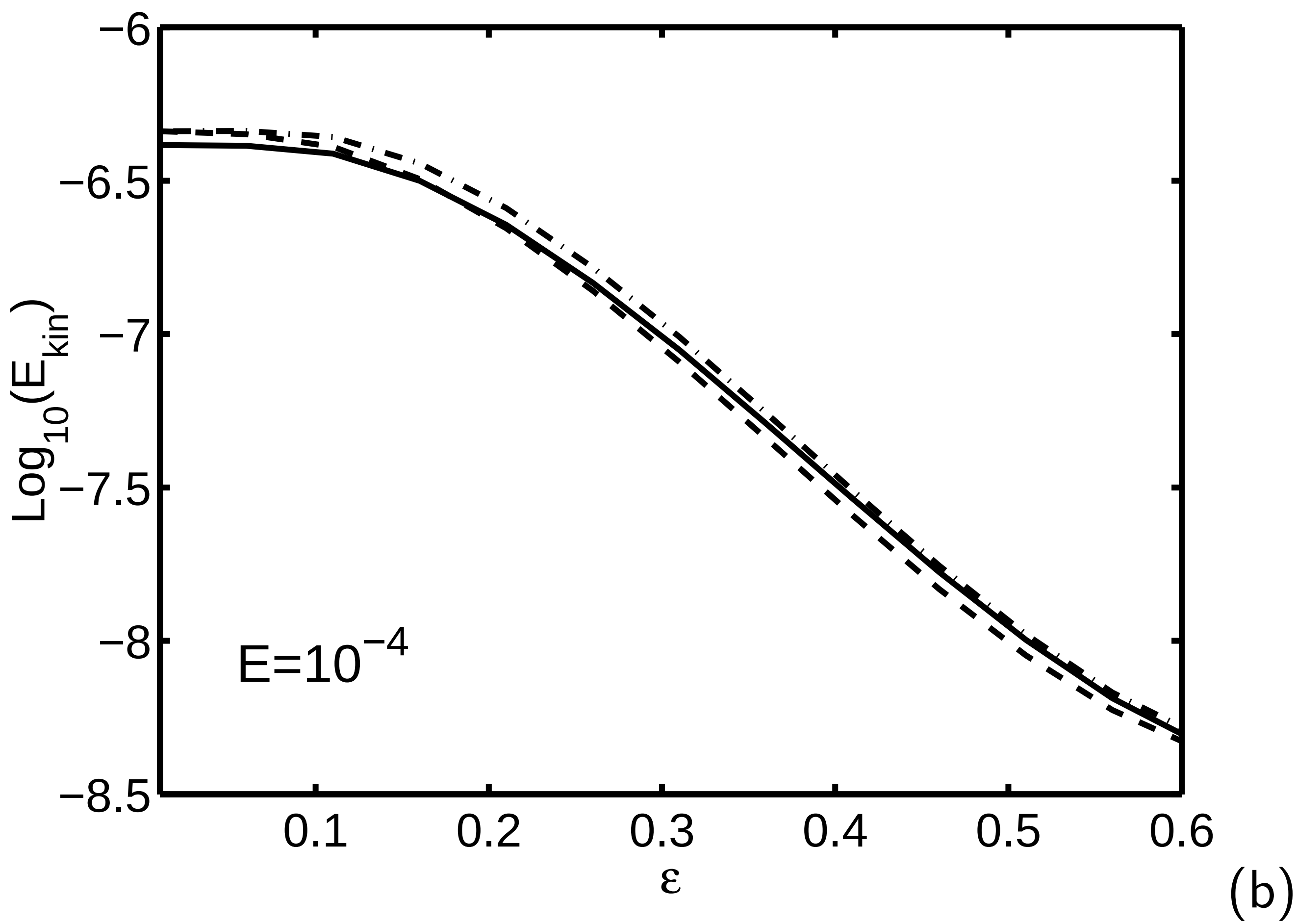


Figure 9: Kinetic energies, $E_{\rm kin}$, of the retrogradely precessing flow as a function of $\mathcal{E}$ for $E = 10^{-4}$, $\alpha = 23.5^o$ and $Po = -10^{-2}$.

## Remarks

• A unified asymptotic approach to the theory of rotating flows is based on the existence of mathematically explicit and relatively simple/complete analytical solutions of a rotating system.

• A unified asymptotic approach to the theory of rotating flows can be readily extended to the compressible fluid in rapidly rotating systems.

• Reference: Zhang, K. and X. Liao (2017) THEORY AND MODELING OF ROTATING FLUIDS, (19 Chapters, $\sim$ 520 pages, ISBN 978-0-521-85009-4), a Cambridge University Monograph on Mechanics, Cambridge University Press.