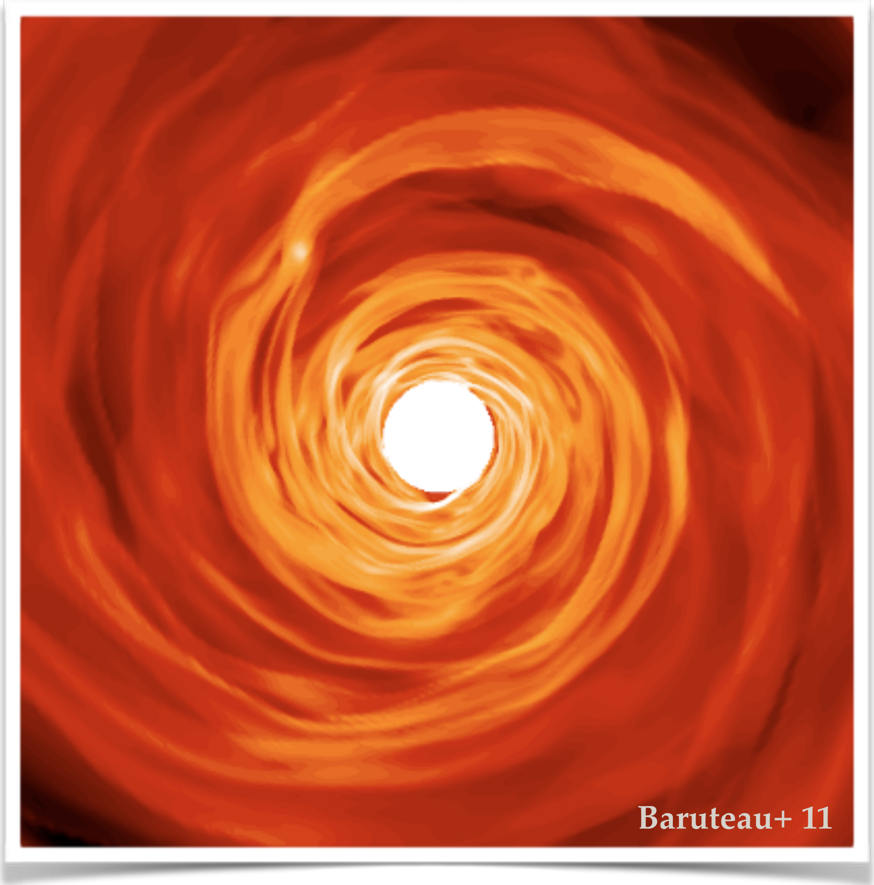
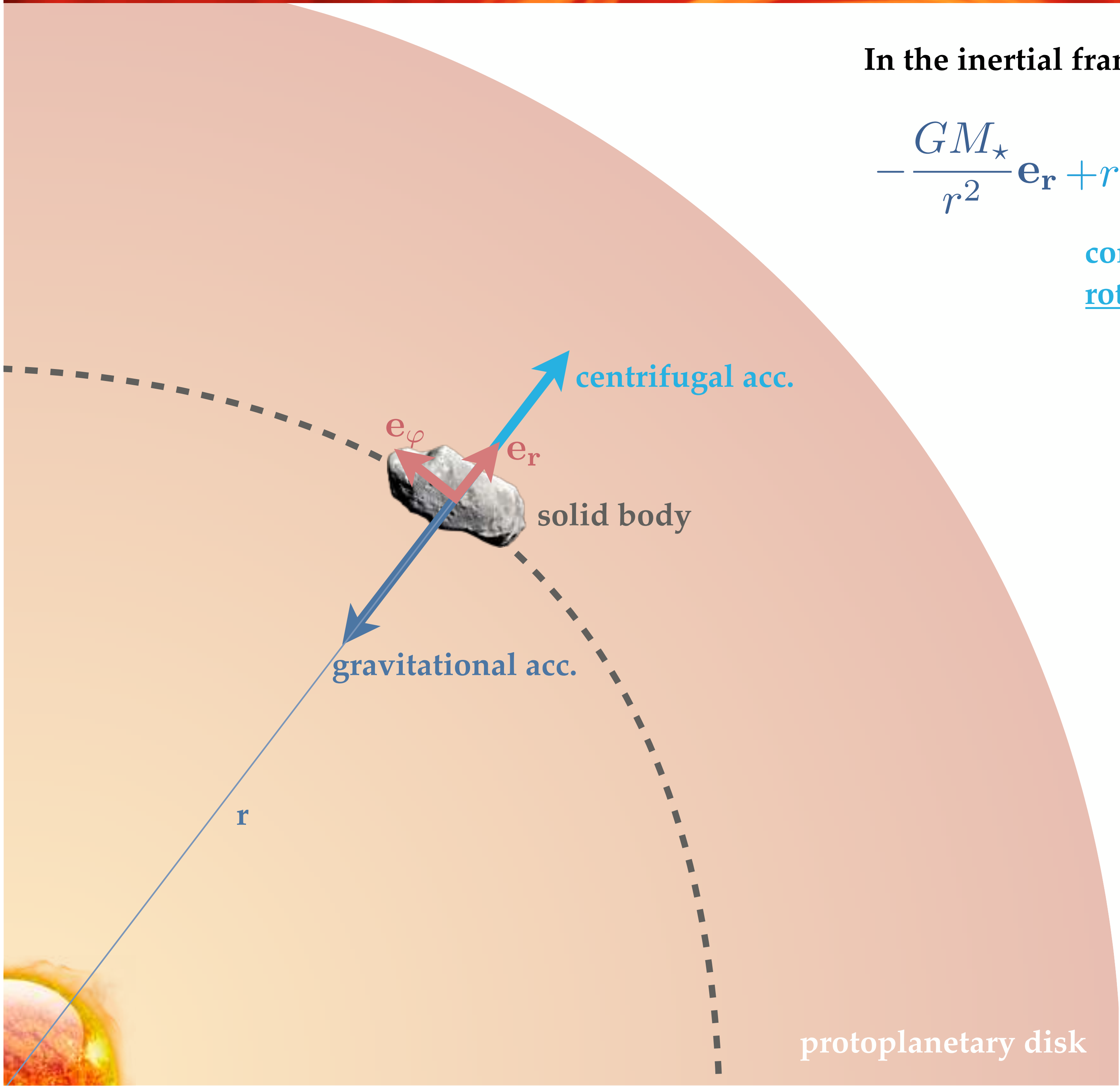
Lecture 2: Gas dynamics in protoplanetary disks



Suggested references:

- Armitage 2011, ***Dynamics of protoplanetary disks*** arxiv.org/abs/1011.1496
- Turner et al. 2014, ***Transport and Accretion in Planet-Forming Disks*** arxiv.org/abs/1401.7306
- Fromang & Lesur 2017, ***Angular momentum transport in accretion disks: a hydrodynamical perspective*** arxiv.org/abs/1705.03319

hydrostatic equilibrium: (i) radial direction

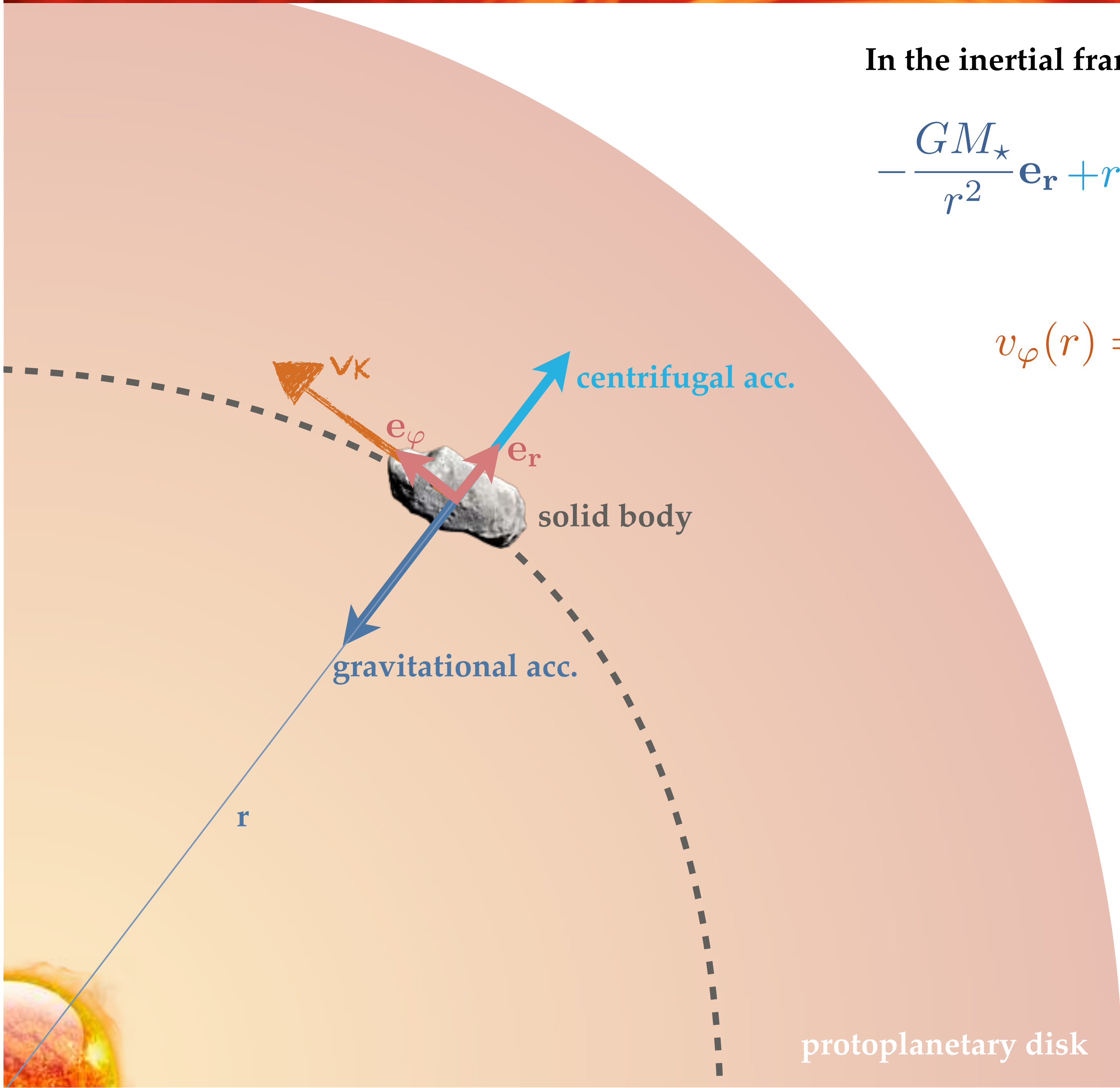


In the inertial frame centred on the star:

$$-\frac{GM_\star}{r^2}\mathbf{e_r} + r\Omega^2\mathbf{e_r}$$

comes about as we're using a
rotating coordinate system

hydrostatic equilibrium: (i) radial direction



In the inertial frame centred on the star:

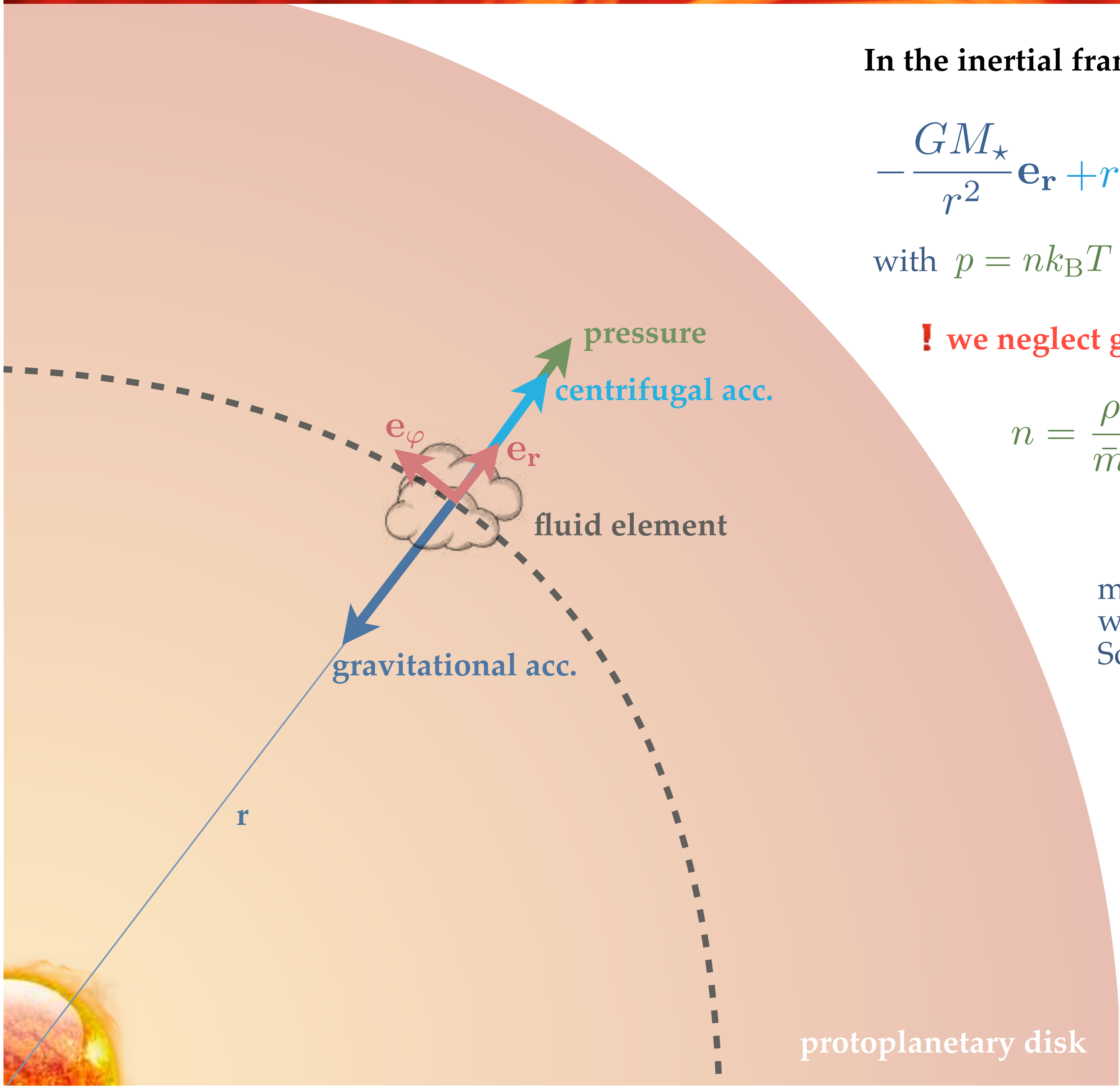
$$-\frac{GM_\star}{r^2}\mathbf{e_r} + r\Omega^2\mathbf{e_r} = \mathbf{0}$$

$$\Downarrow$$

$$v_\varphi(r) = r\Omega = \sqrt{\frac{GM_\star}{r}} \equiv v_\mathrm{K}$$

(Keplerian speed)

hydrostatic equilibrium: (i) radial direction



In the inertial frame centred on the star:

$$-\frac{GM_\star}{r^2}\mathbf{e_r} + r\Omega^2\mathbf{e_r} - \frac{1}{\rho}\frac{\partial p}{\partial r}\mathbf{e_r} = \mathbf{0}$$

with $p = nk_\mathrm{B}T$ the **thermal** pressure (ideal gas law)

❗ **we neglect gas self-gravity, magnetic fields…**

$$n = \frac{\rho}{\bar{m}}$$

← mean mass of gas particles: $\bar{m} = \mu m_\mathrm{p}$

mean molecular weight (μ ~ 2.4 for Solar composition)

proton mass

$$\Rightarrow \boxed{p = \frac{\rho k_\mathrm{B} T}{\mu m_\mathrm{p}} = \rho c_\mathrm{s}^2}$$

(isothermal) sound speed

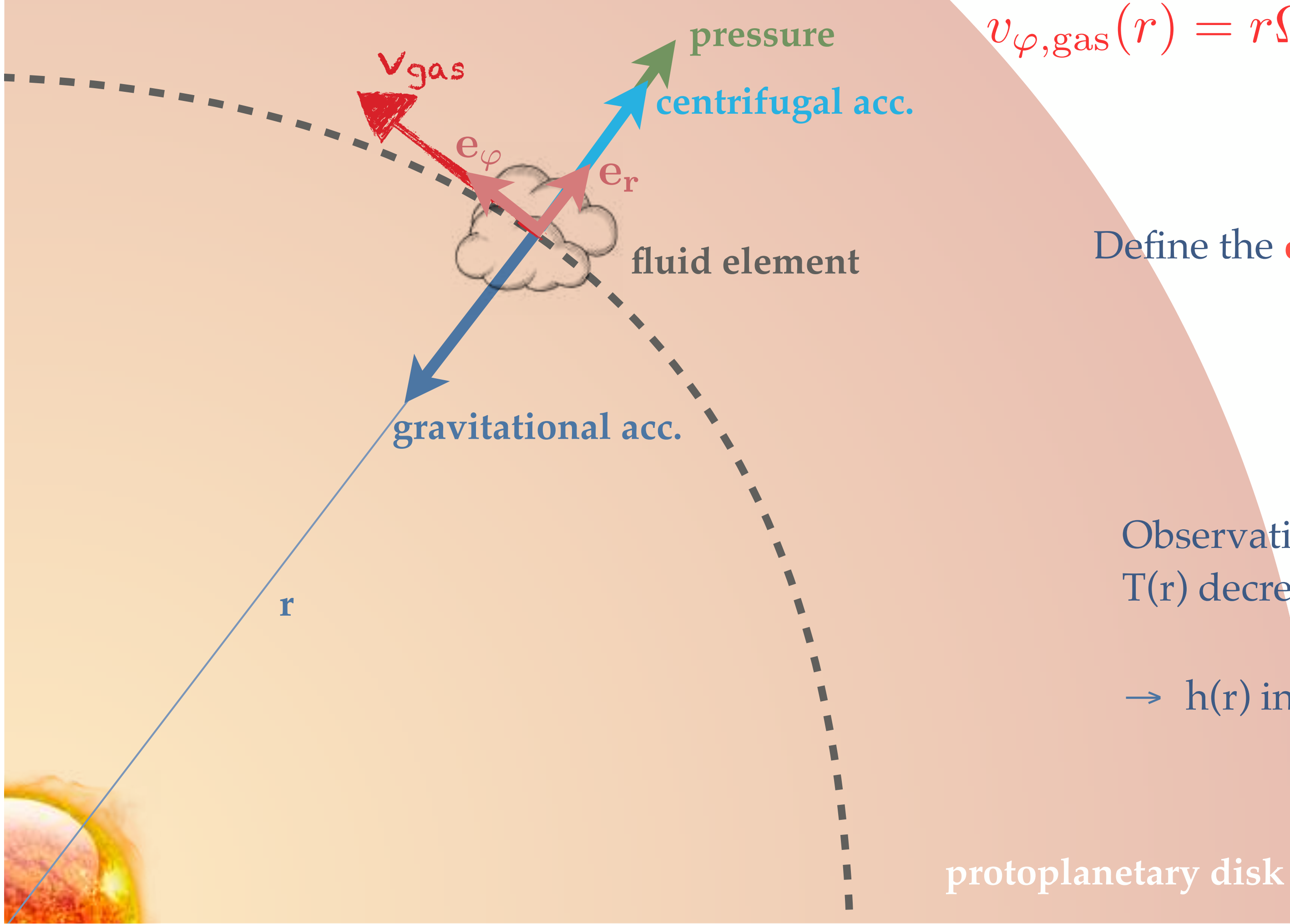
hydrostatic equilibrium: (i) radial direction

$$-\frac{GM_\star}{r^2}\mathbf{e_r} + r\Omega^2\mathbf{e_r} - \frac{1}{\rho}\frac{\partial p}{\partial r}\mathbf{e_r} = \mathbf{0}$$

$$p = \frac{\rho k_\mathrm{B} T}{\mu m_\mathrm{p}} = \rho c_\mathrm{s}^2$$

$$\Downarrow$$

$$v_{\varphi,\mathrm{gas}}(r) = r\Omega = v_\mathrm{K}\left(1 + \frac{c_\mathrm{s}^2}{v_\mathrm{K}^2}\underbrace{\frac{\partial \log p}{\partial \log r}}_{<0}\right)^{1/2}$$



Define the **disk's aspect ratio** as $h = c_\mathrm{s}/v_\mathrm{K}$

$$\rightarrow \quad h = \sqrt{\frac{k_\mathrm{B}T}{\mu m_\mathrm{p}} \times \frac{r}{GM_\star}}$$

Observations and theory indicate that $T(r)$ decreases as $r^{-\beta}$ with β in [1/2 - 3/4]

→ $h(r)$ increases as r^{f} with f in [1/8 - 1/4]

hydrostatic equilibrium: (i) radial direction

$$-\frac{GM_\star}{r^2}\mathbf{e_r} + r\Omega^2\mathbf{e_r} - \frac{1}{\rho}\frac{\partial p}{\partial r}\mathbf{e_r} = \mathbf{0}$$

$$p = \frac{\rho k_B T}{\mu m_p} = \rho c_s^2$$

$$\Downarrow$$

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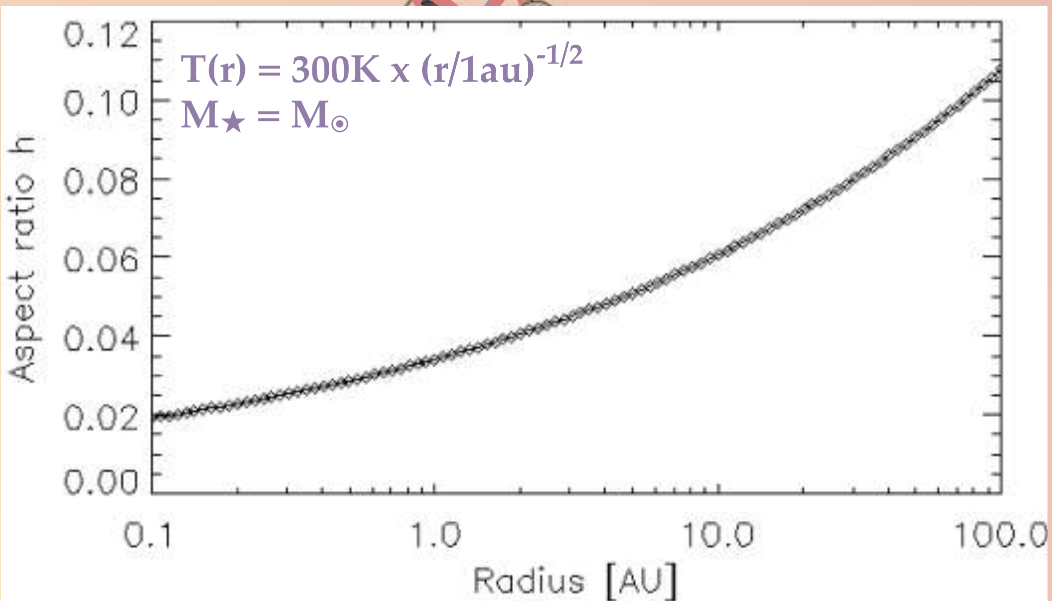

Observations and theory indicate that T(r) decreases as $r^{-\beta}$ with β in [1/2 - 3/4]

$\rightarrow$ h(r) increases as r^f with f in [1/8 - 1/4]

protoplanetary disk

hydrostatic equilibrium: (i) radial direction

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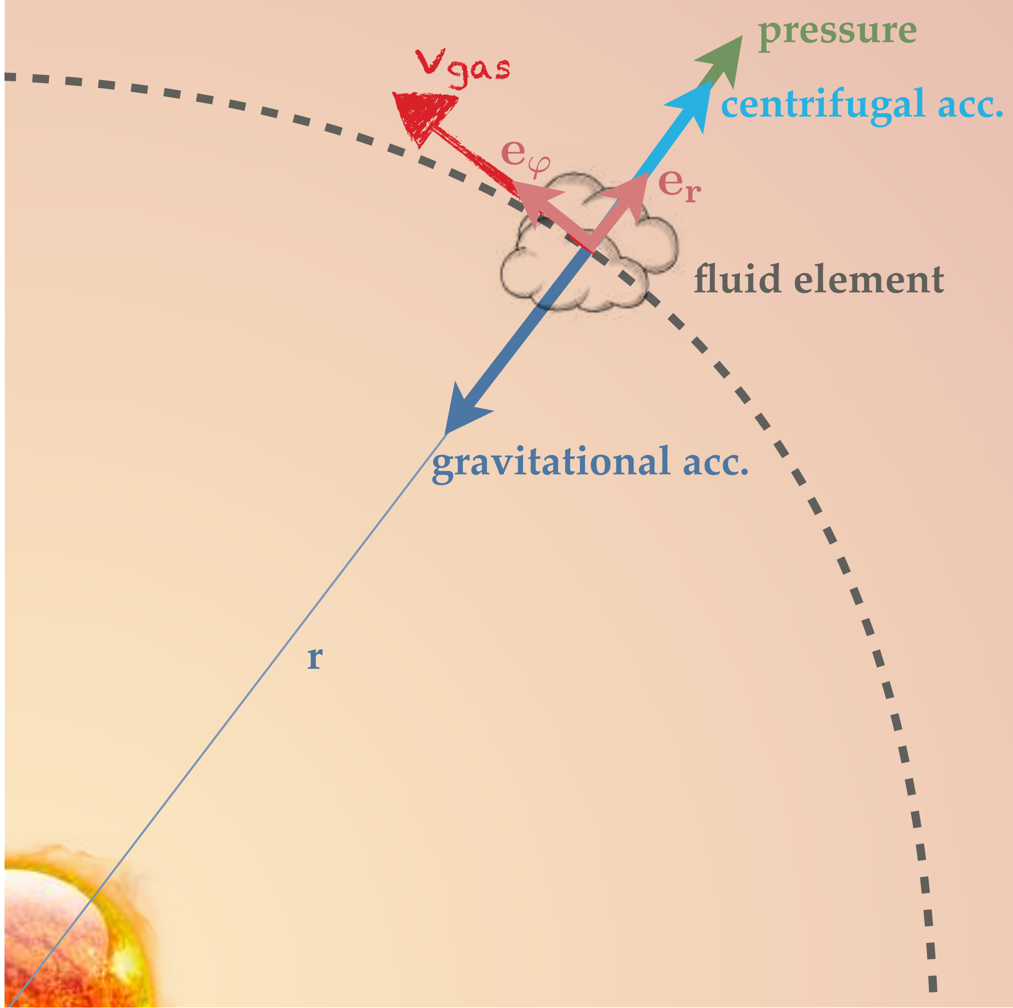
$$p = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{p}}} = \rho c_{\mathrm{s}}^2$$

$$\Downarrow$$

$$v_{\varphi,\mathrm{gas}}(r) = r\Omega = v_{\mathrm{K}}\left(1 + \frac{c_{\mathrm{s}}^2}{v_{\mathrm{K}}^2}\frac{\partial \log p}{\partial \log r}\right)^{1/2}$$

$[10^{-3}\text{-}10^{-2}] \ll 1$

< 0

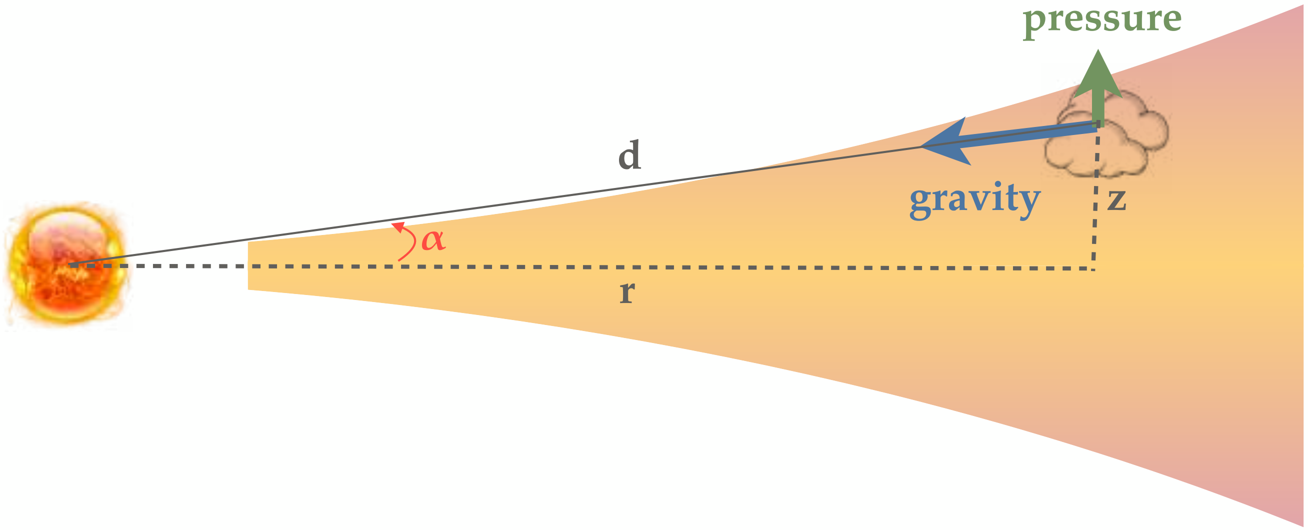


protoplanetary disk

→ **gas is (very) slightly sub-Keplerian**

<u>NB</u>: magnetic fields and self-gravity hardly alter this picture

hydrostatic equilibrium: (ii) vertical direction



$$-\frac{GM_\star}{d^2}\sin\alpha\mathbf{e_z} - \frac{1}{\rho}\frac{\partial p}{\partial z}\mathbf{e_z} = \mathbf{0}$$

discard again self-gravity, B fields and assume the disk is thin (z « R):

$$\rightarrow \frac{1}{\rho}\frac{\partial p}{\partial z} \approx -\frac{GM_\star}{r^3} \times z = -\Omega_K^2 z$$

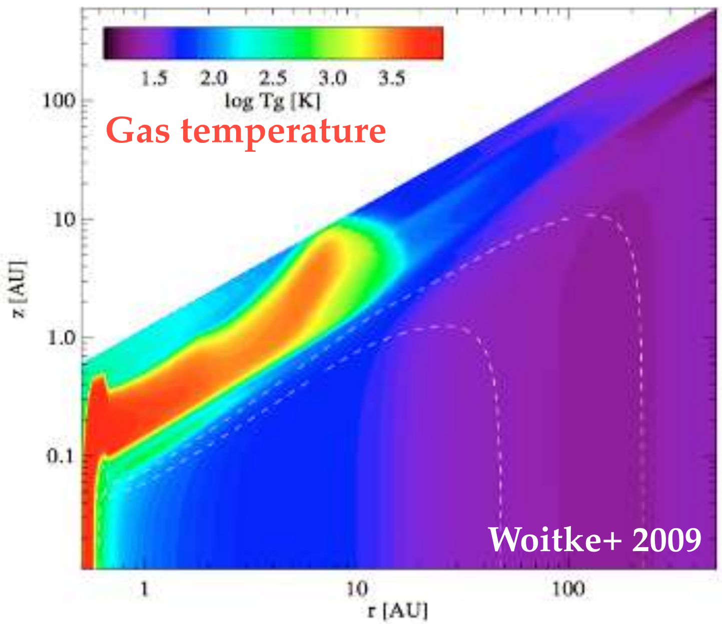
Now recall that $p = \rho c_s^2$ with both ρ and c_s (hence temperature) functions of r and z, a priori

Progress can be made by assuming the disk is **vertically isothermal** which is not too bad an assumption near the disk midplane, where most of the mass is contained → $c_s = c_s(r)$

We obtain: $$\frac{c_s^2(r)}{\rho(r,z)}\frac{\partial\rho}{\partial z} = -\Omega_K^2 z \rightarrow \rho(r,z) = A(r)\exp\left(-\frac{z^2\Omega_K^2}{2c_s^2}\right)$$

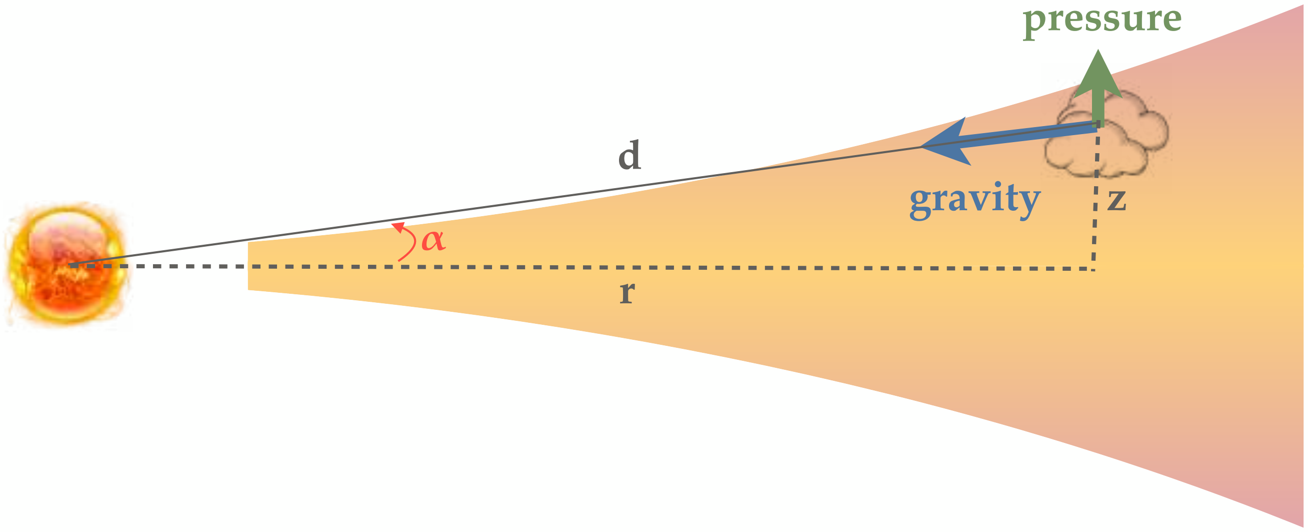
Finally, $$\rho(r,z) = \rho(r, z=0)\exp\left(-\frac{z^2}{2H^2}\right)$$

with $H = c_s/\Omega_K$ the disk's pressure scale height



radiation thermo-chemical disk model

hydrostatic equilibrium: (ii) vertical direction



$$\rho(r,z) = \rho(r,z=0)\exp\left(-\frac{z^2}{2H^2}\right)$$

with $H = c_s/\Omega_K$

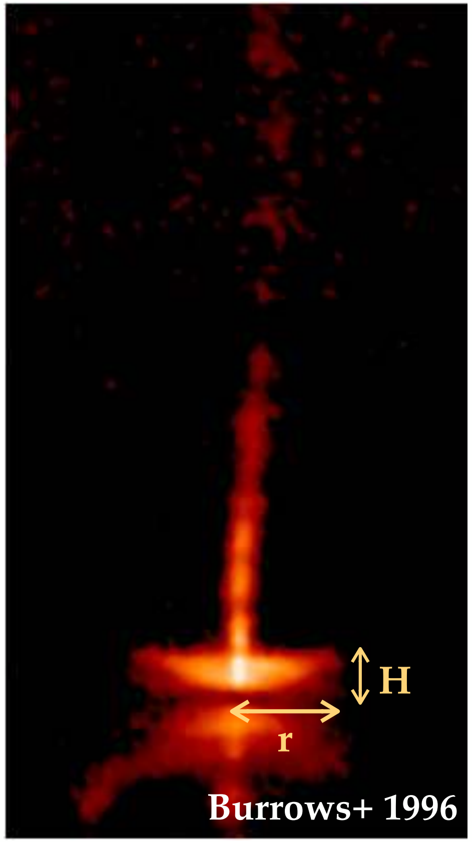
- We note that $H = h \times r$ with $h = c_s/v_K$ the disk's aspect ratio

  → H(r) increases faster than linearly with r: disks are said to be **flared** which is thought to explain the bowl shape of edge-on disks (though what we see is dust and not gas…)

- Note also that the gas **surface density** $\Sigma(r) = \int_{-\infty}^{\infty} \rho(r,z)dz$

  → $\rho(r,z=0) = \frac{\Sigma}{\sqrt{2\pi}H} \sim \frac{\Sigma}{2H}$

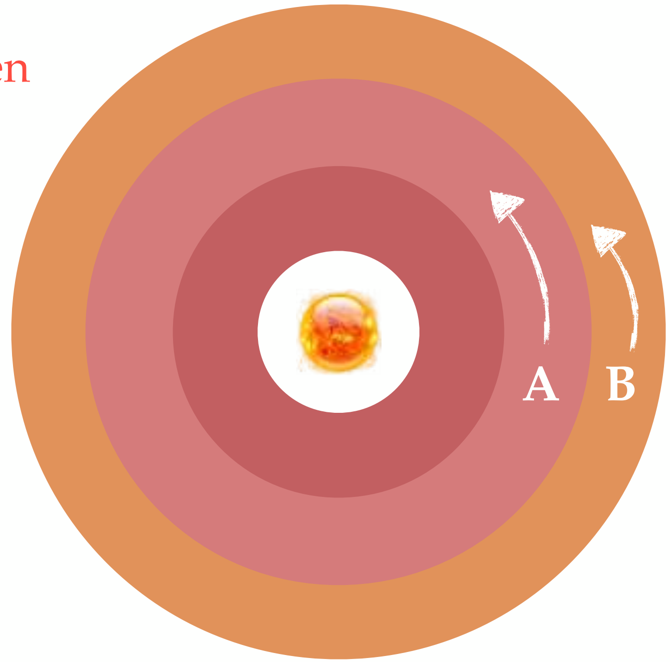
  typical numbers at r=1au: $H/r \sim 0.03, \Sigma \sim 10^3\,\mathrm{g\,cm^{-2}}, \rho \sim 10^{-9}\,\mathrm{g\,cm^{-3}}$



How is the disk accreted then?

- The disk **cannot be** in **strict centrifugal balance**, otherwise gas will stay on stable circular orbits forever and will never be accreted onto the central star!

💡 **differential rotation** implies **viscous friction** forces between neighboring rings of gas:



- ring A moves faster than ring B: friction between the two rings will try to slow down A and speed up B

  → **angular momentum** is **transferred** from A to B

specific angular momentum for a Keplerian disk:

$$l = r v_\varphi = r^2 \Omega_\mathrm{K} = \sqrt{G M_\star r}$$

- so, if ring A **loses angular momentum**, but has to remain on a **Keplerian** orbit, it must **move inward**! Ring B then moves outward, unless it has friction too with a ring C (which has friction with another ring D, etc…)

- in brief: through **radial** transport by viscosity, the **inner disk falls in**, the **outer disk expands**

How is the disk accreted then?

- The disk **cannot be** in **strict centrifugal balance**, otherwise gas will stay on stable circular orbits forever and will never be accreted onto the central star!

differential rotation implies **viscous friction** forces between neighboring rings of gas

- But what is the **molecular viscosity** in protoplanetary disks?

From gas kinetic theory, we know that the gas kinematic viscosity is $\nu \sim \lambda c_s$

with the **mean-free path** of the gas molecules $\lambda \sim \frac{1}{n\sigma} = \frac{\mu m_p}{\rho\sigma}$

$\rho \sim \Sigma/2H$ **cross section ~ π x diameter²**

Plugging in typical numbers at 1 au, we find $\nu \sim 10^2\ m^2\ s^{-1}$ …

… so that the **viscous timescale** at R=1 au is **~R^2/ν ~ 3×10^{12} yr!** » **disks lifetime!**

→ molecular viscosity cannot explain accretion in protoplanetary disks!

- Define **Reynolds number** with fluctuating velocity scale **~c_s** and corresponding length scale **~H**

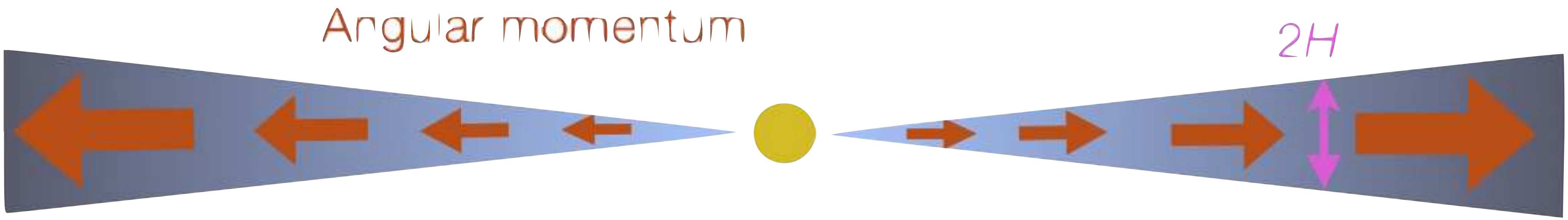
at 1 au, **Re = $c_s H/\nu$ ~ 10^{10}** ! → protoplanetary disks are likely turbulent

question is: via linear or non-linear instabilities?

How is the disk accreted then?

- The disk **cannot be** in **strict centrifugal balance**, otherwise gas will stay on stable circular orbits forever and will never be accreted onto the central star!

  turbulent (radial) transport of angular momentum due to MHD **instabilities**



G. Lesur

- **IF** turbulent transport is associated with **local energy dissipation**, it can be **modelled** as a viscous diffusion process

Balbus & Papaloizou 99

→ alpha disk model: $\boxed{\nu = \alpha c_s H}$ with $\alpha < 1$

Shakura & Sunyaev 73

interpretation: viscosity is due to **turbulent eddies** with **mean free path** ≲ **H** and **speed** ≲ c_s

→ to explain disks lifetime and measured (stellar) accretion rates, we need $\alpha \sim [10^{-3}-10^{-2}]$

- **many** instabilities have been investigated over the last decades, in this lecture we will only go through some of them!

How is the disk accreted then?

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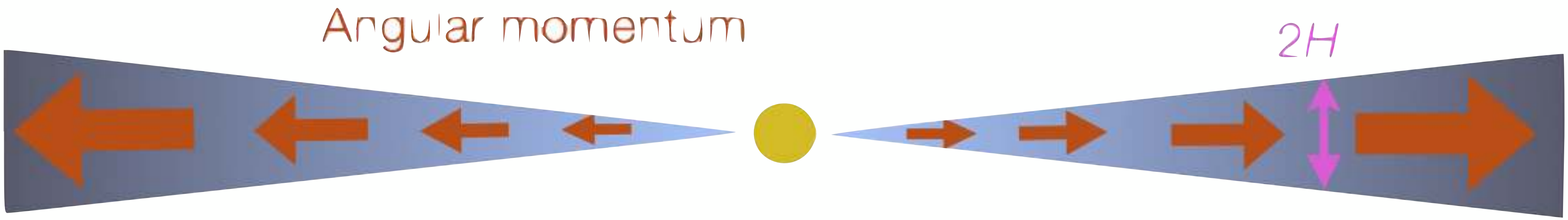
💡 **turbulent (radial) transport** of angular momentum due to MHD **instabilities**



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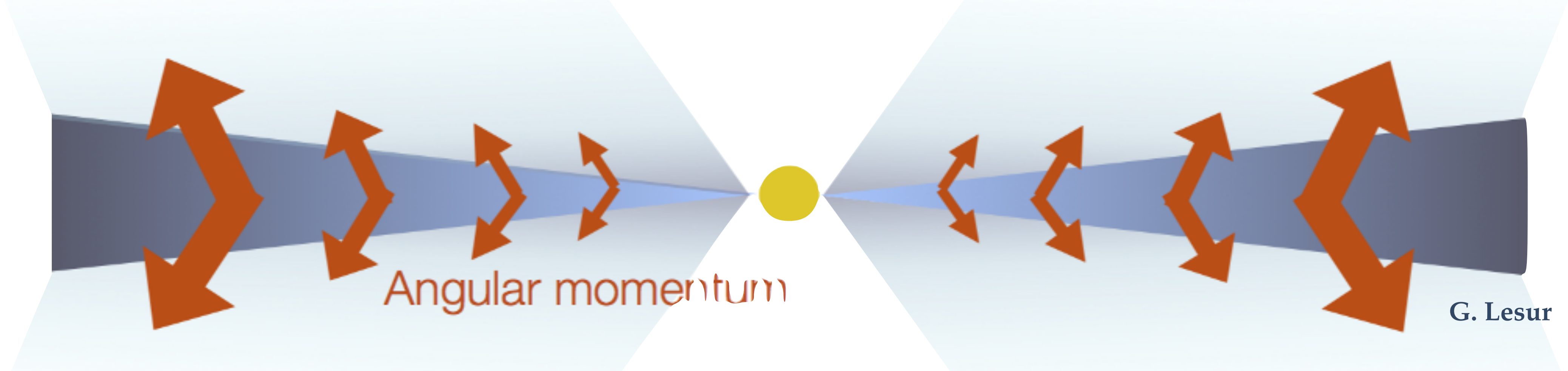
💡 **vertical transport (extraction)** of angular momentum by **magneto-centrifugal winds**

→ **vertical magnetic field** exerts a **torque** on the disk **surface** which implies the entire disk surface falls in (not a viscous diffusion process!) e.g., Blandford & Payne 82

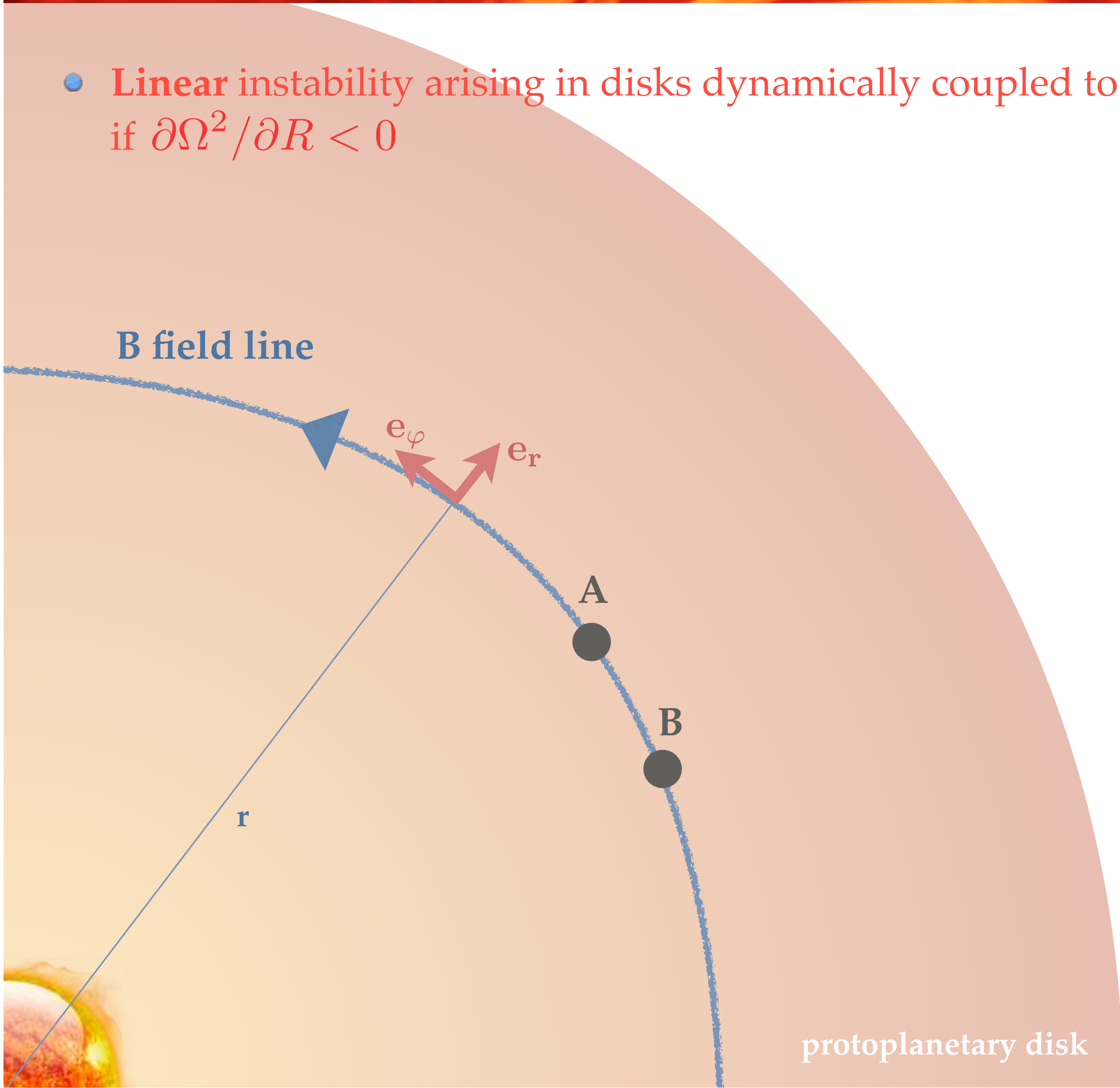


G. Lesur

Magneto-rotational instability (MRI)

- **Linear** instability arising in disks dynamically coupled to a weak **magnetic** field if $\partial\Omega^2/\partial R < 0$

Balbus & Hawley 91, Balbus 03



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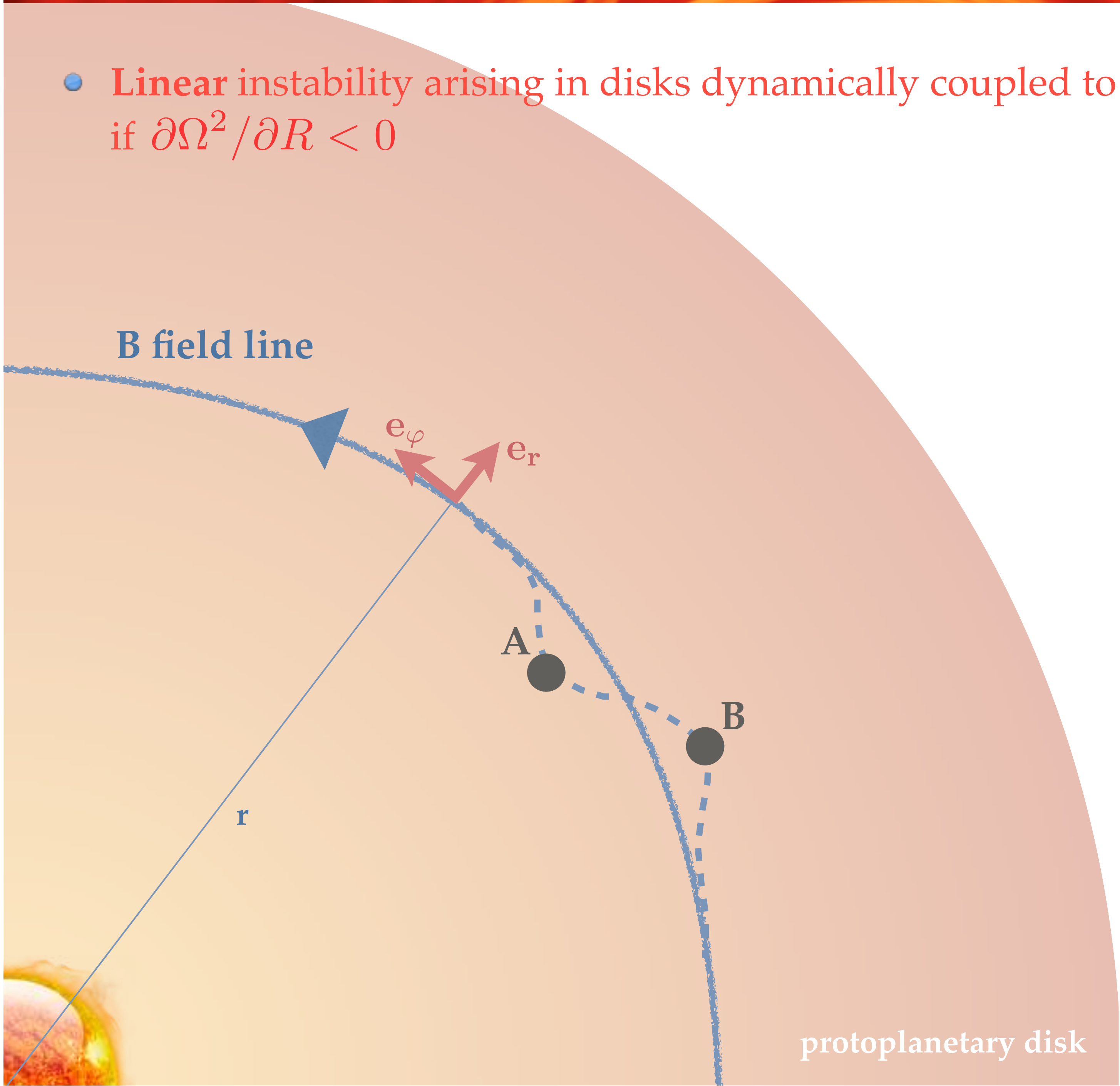

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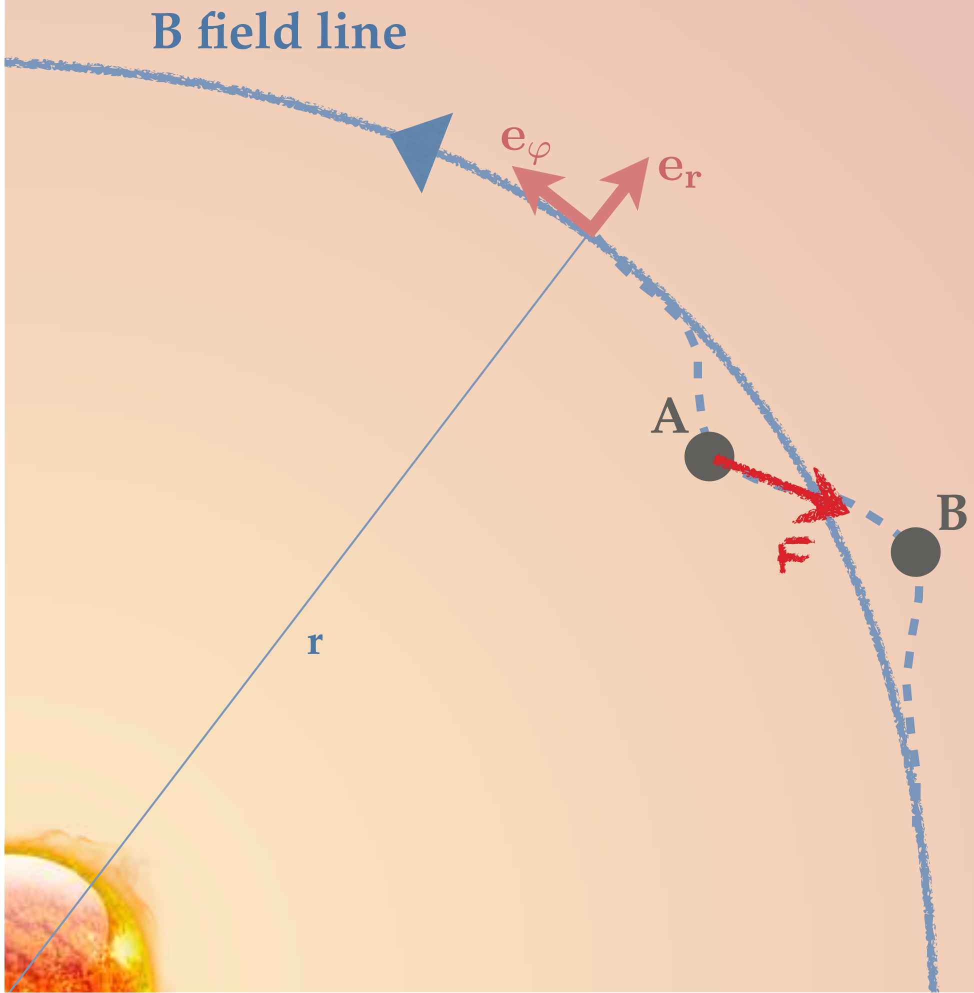

Torque on A due to magnetic tension $\Gamma \sim r_A \times F_\phi < 0$

$\rightarrow$ A's specific angular momentum (j) decreases (Γ=dj/dt)

$\rightarrow$ A moves further in! ($j = rv_\varphi = \sqrt{GM_\star r}$)

protoplanetary disk

Magneto-rotational instability (MRI)

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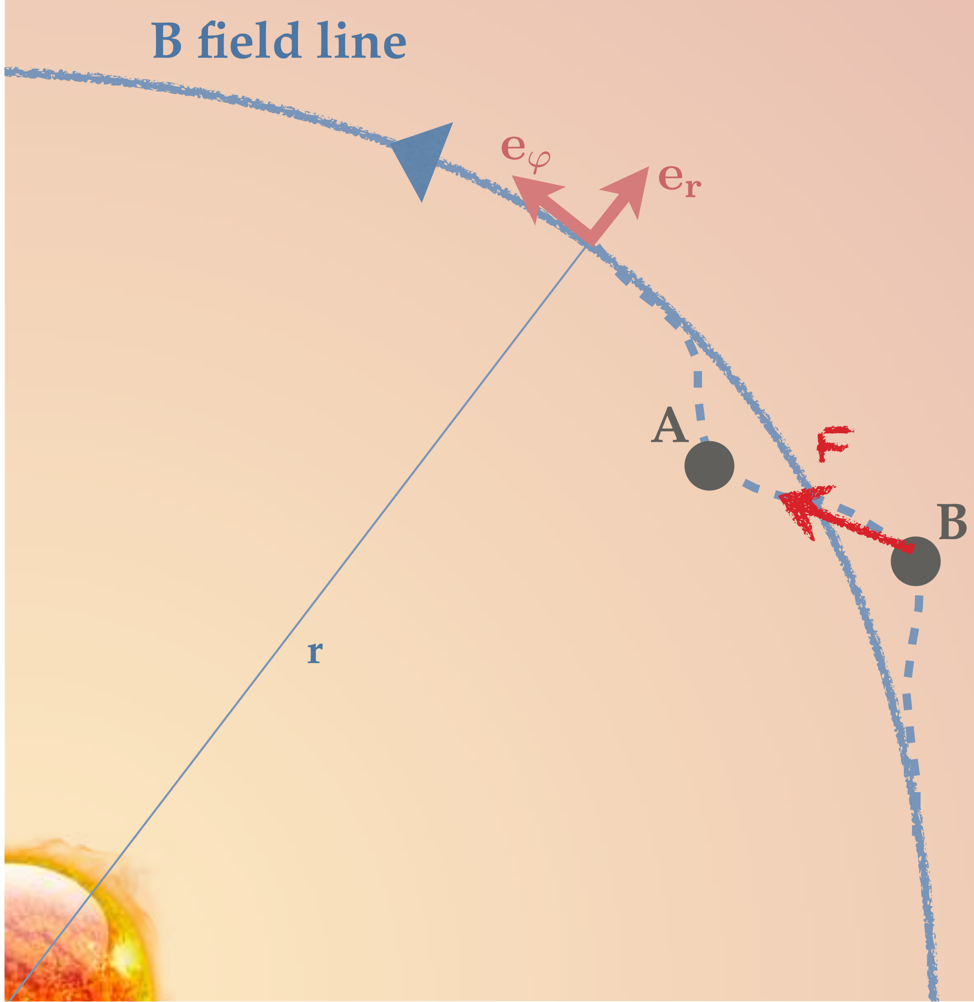
Balbus & Hawley 91, Balbus 03



Torque on B due to **magnetic tension** $\Gamma \sim r_B \times F_\phi > 0$

→ B's specific angular momentum (j) increases (Γ=dj/dt)

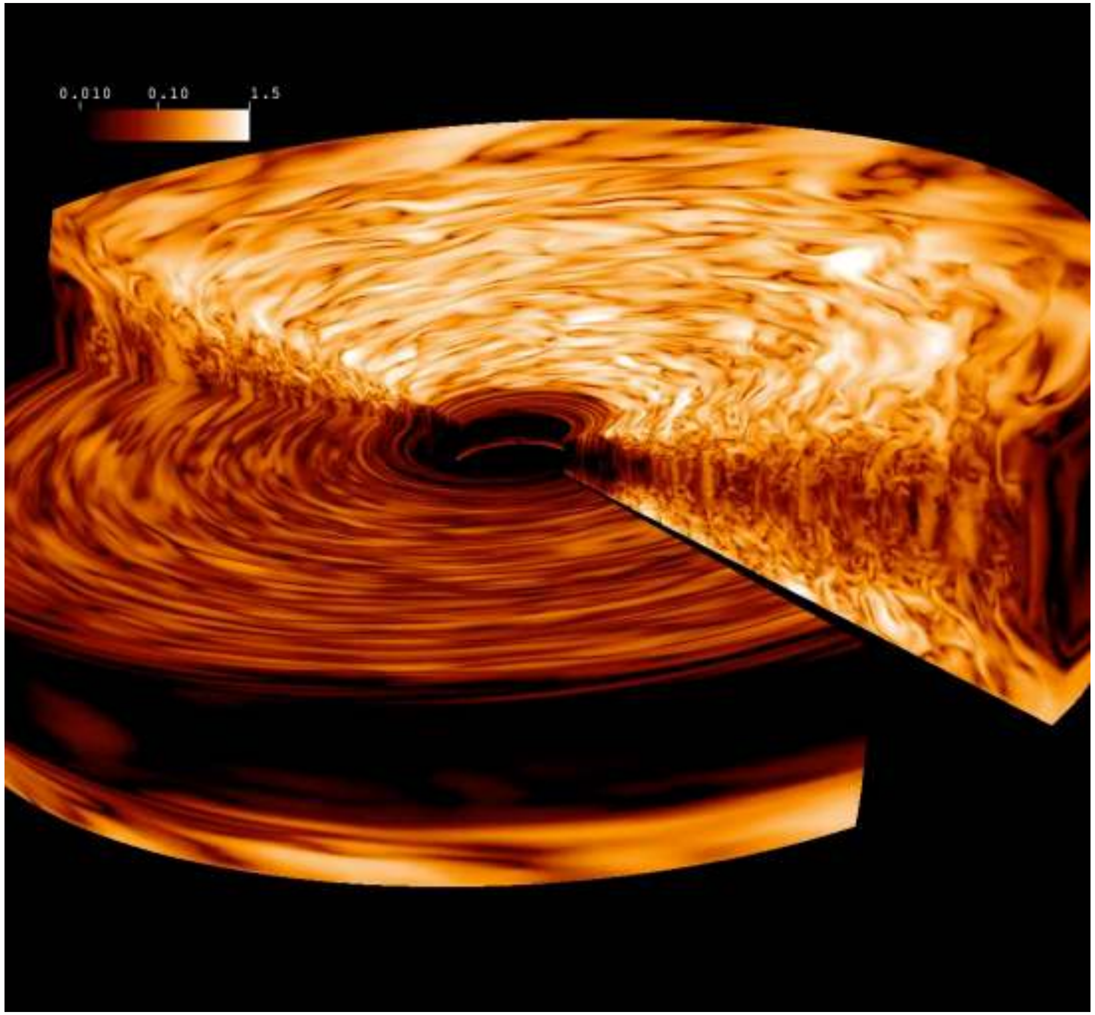
→ B moves further out! ($j = rv_\varphi = \sqrt{GM_\star r}$)

Magneto-rotational instability (MRI)

- Linear instability arising in disks dynamically coupled to a **weak** magnetic field if $\partial\Omega^2/\partial R < 0$

Balbus & Hawley 91, Balbus 03

$$|B|^2/2\mu_0 \lesssim \rho c_s^2$$



Gas Mach number (r.m.s. turbulent velocity in units of the local sound speed). Disk extends from R=0.5 to 1.5 AU, and the r.m.s. turbulent velocity goes from ~1 to ~1000 m/s

Flock+ 2013

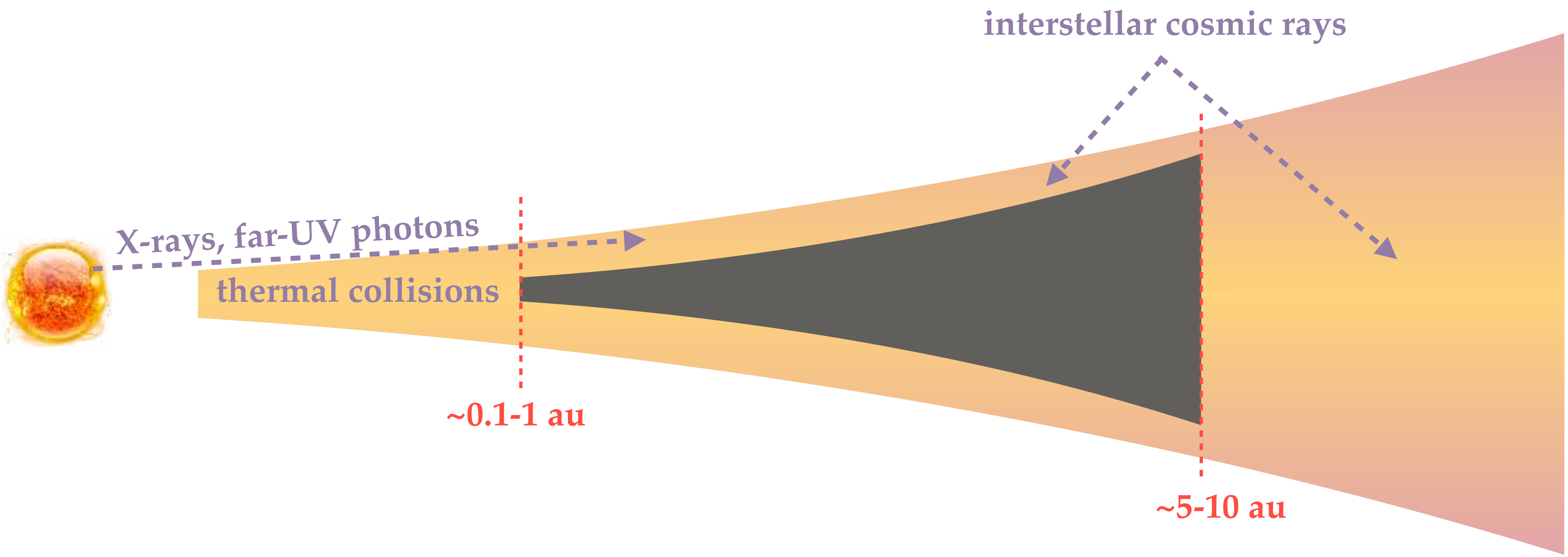
→ the disk reaches a quasi steady-state with **turbulent** mass **accretion** rates in fair agreement with observations (α ~ $[10^{-3}$-$10^{-2}]$)

Magneto-rotational instability (MRI)

- Linear instability arising in disks **dynamically** coupled to a weak magnetic field if $\partial\Omega^2/\partial R < 0$

Balbus & Hawley 91, Balbus 03

protoplanetary disks are in fact **poorly** ionized! (n_e / $n < 10^{-13}$)



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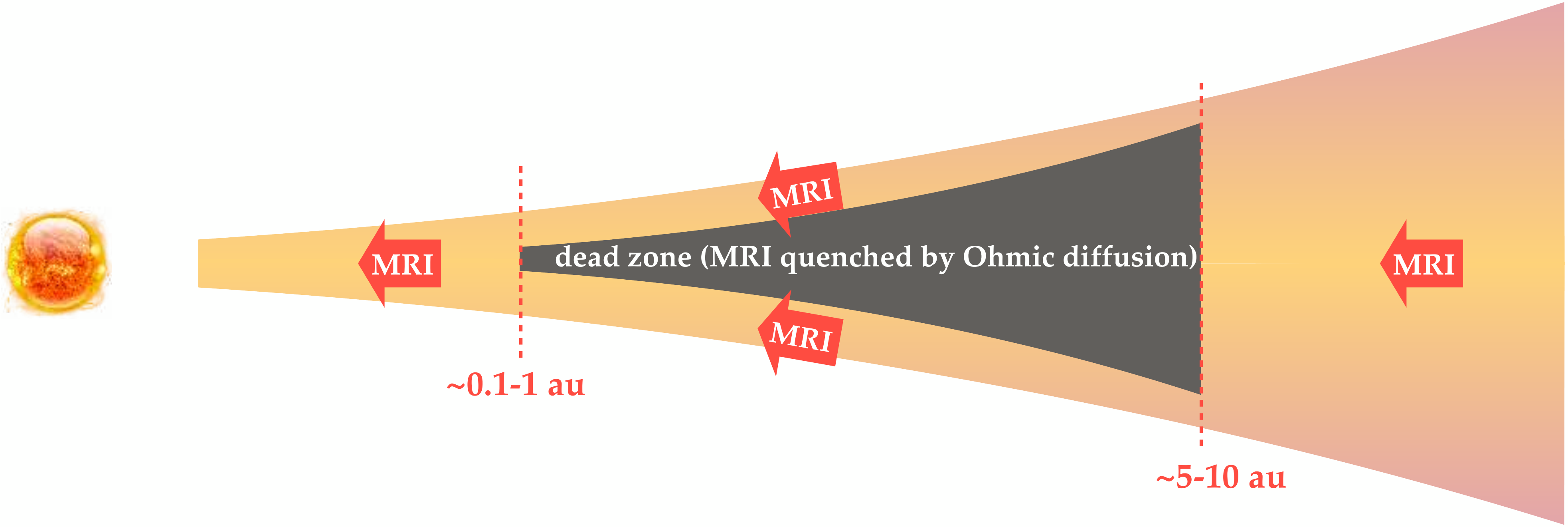

→ Ohmic diffusion (electrons-neutrals collisions) makes a large fraction of the bulk disk magnetically inactive

→ **layered accretion**

Gammie 96

Magneto-rotational instability (MRI)

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Balbus & Hawley 91, Balbus 03

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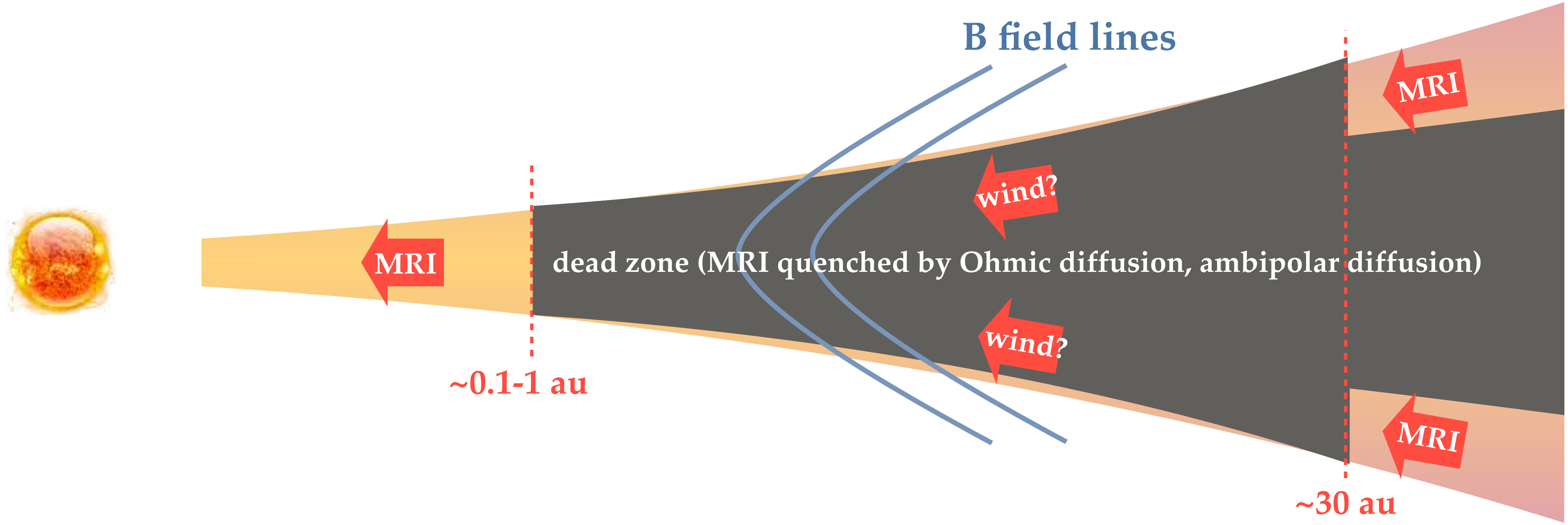


→ Ambipolar diffusion (ions-neutrals collisions) largely **quenches** MRI in the disk's surface layers, and partly in its outer parts

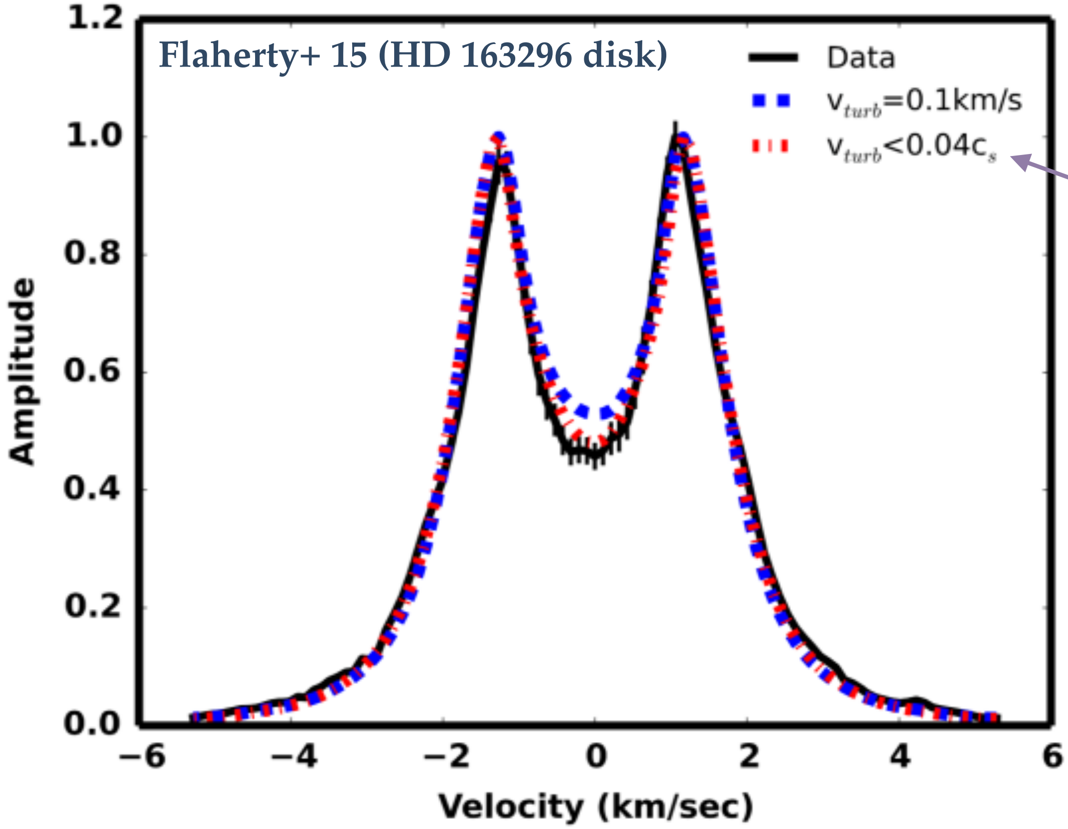
→ **Wind**-driven accretion if a vertical B field threads the disk **Bai 13, Simon+ 13…**

→ very active field of research! Talk by **G. Lesur** next week

Magneto-rotational instability (MRI): take away

- Though it is a potentially powerful source of accretion, **MHD turbulence** due to MRI is **likely absent** in large (1-30 AU) parts of protoplanetary disks

- This is overall consistent with **observations** of the (small!) **non-thermal broadening** of molecular gas lines (e.g., CO gas) in disks



subsonic turbulence is favored in the surface of this disk

Magneto-rotational instability (MRI): take away

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→ what drives accretion in the bulk of protoplanetary disks? Magnetic winds? Turbulence due to **hydrodynamic instabilities?**

→ implications on models of planet formation and evolution

Rossby-wave instability (RWI)

- **Linear** instability driven by a **radial extremum** in the quantity $\mathcal{L} = \frac{\Sigma}{2\omega_z}\left(\frac{P}{\Sigma^\gamma}\right)^{2/\gamma}$ with $\omega_z = (\nabla \times u)\cdot\hat{z}$ the gas vorticity **Lovelace+ 99, Li+ 00, 01…**

  γ: adiabatic index

- Disk analogue of the **barotropic instability** (cf. talks by **P. Read** and **J. Park** yesterday)

- Dispersion relation *analogous* to that of **Rossby waves** in **planetary atmospheres**

$$\widetilde{\omega} = -\frac{2\kappa^2}{\kappa^2 + |k|^2 c_s^2} \times \frac{m c_s^2}{r\Sigma}\frac{d\mathcal{L}}{dr} \qquad \vec{k} = (k_r, m/r)^T$$

Méheut+ 13

κ is the radial epicyclic frequency: $\kappa^2 = \frac{1}{R^3}\frac{dj^2}{dR} \equiv \Omega^2$

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  adiabatic index (of γ)

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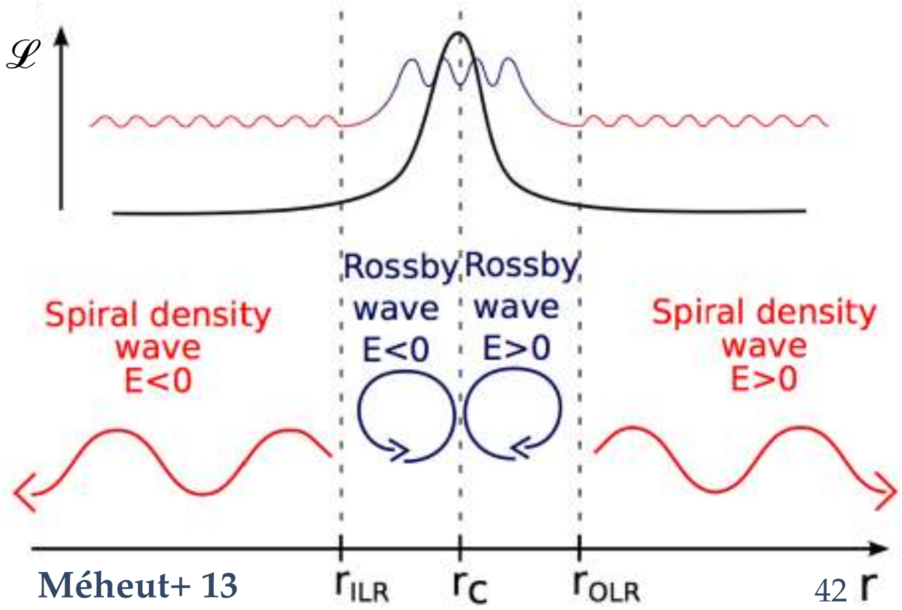
Méheut+ 13

  - A **Rossby wave** propagates on each side of the $\mathscr{L}$ extremum, with fluxes of energy **E** (or angular momentum) of opposite sign

  - Emission of spiral **density waves** (wakes) **beyond** so-called **Lindblad resonances**, where

$$\tilde{\omega}^2 = \Omega^2 + |k|^2 c_s^2$$

  - Instability growth related to **energy exchange between the Rossby waves** — growth rate is sensitive to sound speed, how peaked the $\mathscr{L}$ extremum is, viscosity, inclusion of self-gravity…



Méheut+ 13

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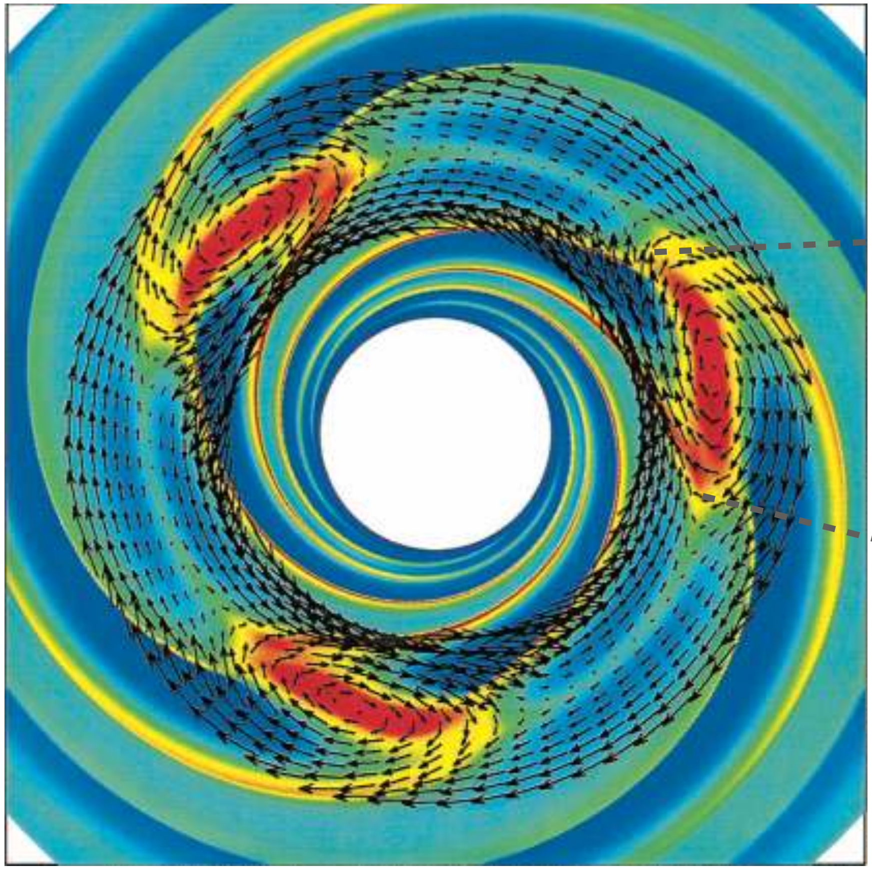
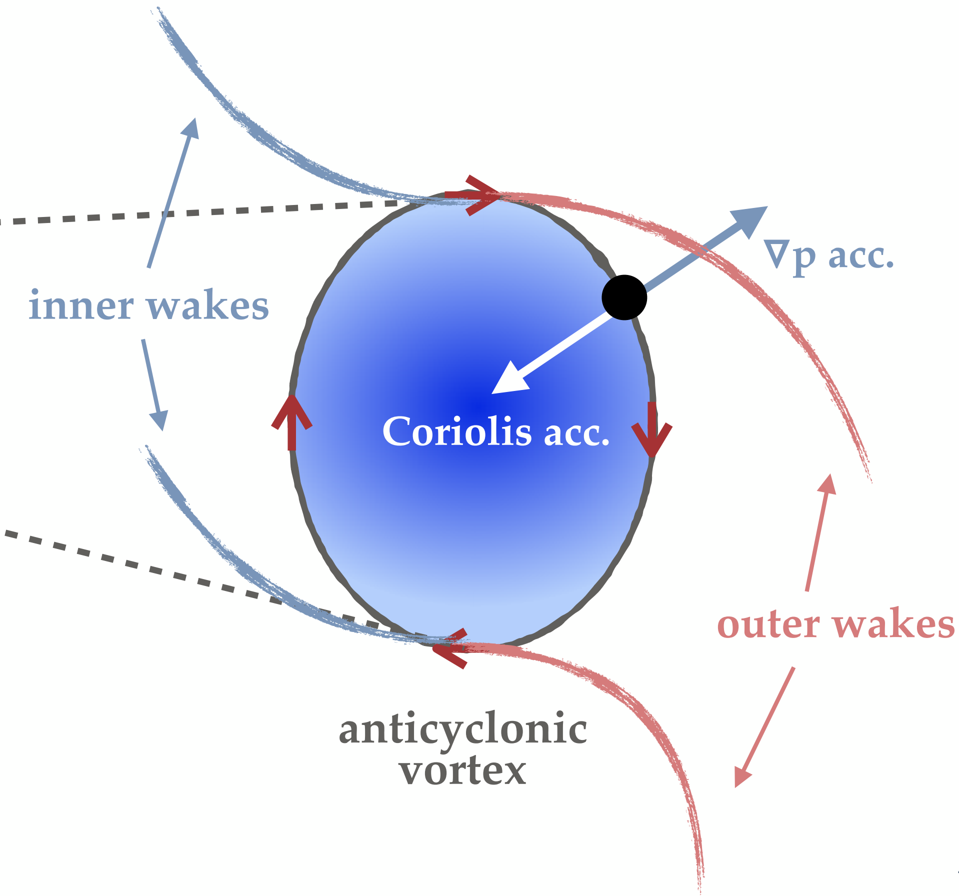
- Disk analogue of the **barotropic instability**

- Saturates into few **anticyclonic vortices** that tend to **merge** in time



disk's perturbed pressure Li+ 01



Rossby-wave instability (RWI)

$$\mathcal{L} = \frac{\Sigma}{2\omega_z}\left(\frac{P}{\Sigma^\gamma}\right)^{2/\gamma}$$

- **Linear** instability driven by a **radial extremum** in the quantity with $\omega_z = (\nabla \times u) \cdot \hat{z}$ the gas vorticity
  Lovelace+ 99, Li+ 00, 01…

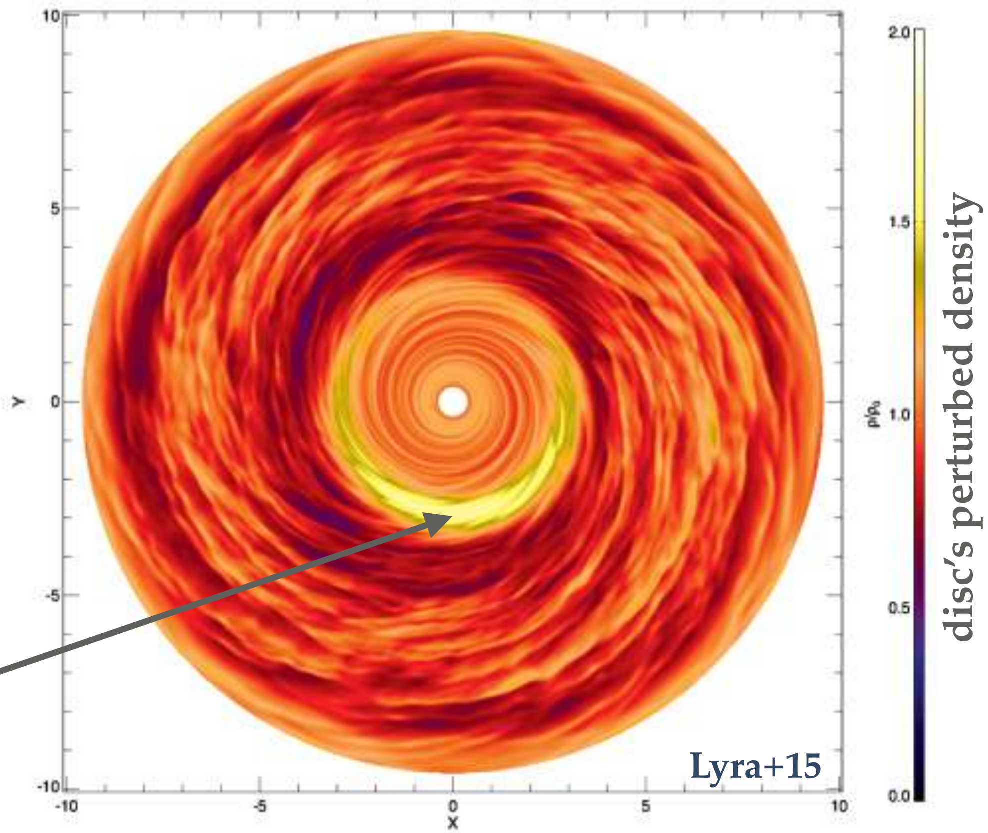
- Disk analogue of the **barotropic instability**

- Saturates into few **anticyclonic vortices** that tend to **merge** in time

- RWI in protoplanetary disks may be **triggered** at:

  → the edges between magnetically active and **dead regions**

  Varnière & Tagger 06, Faure+ 14, Lyra+15…



Vortex between a magnetically dead inner region and active outer region

Rossby-wave instability (RWI)

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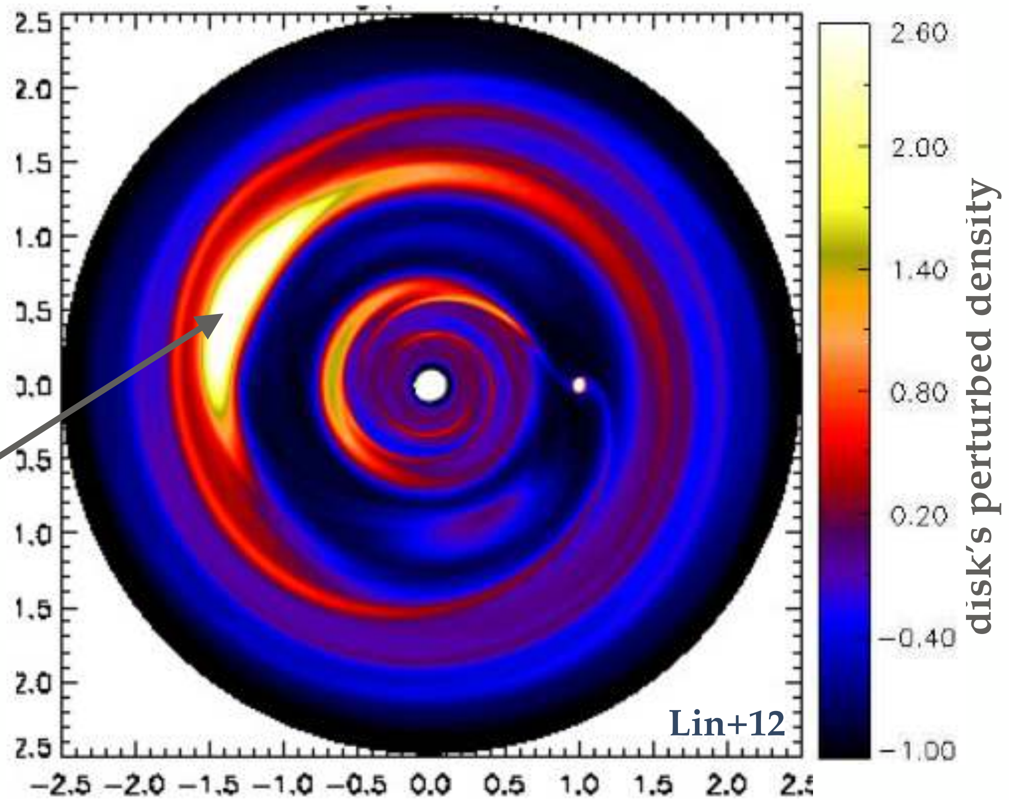
$$\mathcal{L} = \frac{\Sigma}{2\omega_z}\left(\frac{P}{\Sigma^\gamma}\right)^{2/\gamma}$$

Lovelace+ 99, Li+ 00, 01…

- Disk analogue of the **barotropic instability**
- Saturates into few **anticyclonic vortices** that tend to **merge** in time
- RWI in protoplanetary disks may be **triggered** at:

→ the edges of a **planet gap**

de Val Borro+ 07, Lyra+ 09, Lin & Papaloizou 11…



Vortex at the outer edge of a gap carved by a Jupiter-mass planet in an inviscid disk

Rossby-wave instability (RWI)

- **Linear** instability driven by a **radial extremum** in the quantity $\mathcal{L} = \frac{\Sigma}{2\omega_z}\left(\frac{P}{\Sigma^\gamma}\right)^{2/\gamma}$ with $\omega_z = (\nabla \times u)\cdot\hat{z}$ the gas vorticity Lovelace+ 99, Li+ 00, 01…

- Disk analogue of the **barotropic instability**

- Saturates into few **anticyclonic vortices** that tend to **merge** in time

- RWI in protoplanetary disks may be **triggered** at the edges between magnetically active and dead regions, or at the edges of planet gaps

- Evolution of vortices sensitive to the presence of other sources of **turbulence** in the disk (often modelled as a **viscosity**), of **dust** trapped in the vortices etc.

- RWI-induced vortices often invoked to explain the **asymmetric continuum** emission **observed** in disks at radio λ (see lecture 3)

Temperature-related instabilities

- **Preamble**: ***sufficient*** condition for a non-magnetized sheared flow to be **stable** against infinitesimal, axisymmetric, adiabatic perturbations is given by the **Solberg-Høiland criterion**:

Tassoul 78

$$\frac{1}{R^3}\frac{\partial j^2}{\partial R} - \frac{1}{C_{\rm p}\rho}\nabla P \cdot \nabla S > 0 \quad \textbf{[1]} \qquad\qquad \frac{\partial P}{\partial z}\left(\frac{\partial j^2}{\partial R}\frac{\partial S}{\partial z} - \frac{\partial j^2}{\partial z}\frac{\partial S}{\partial R}\right) < 0 \quad \textbf{[2]}$$

with $j = R^2\Omega$ the specific angular momentum and $S = S_0 + C_v \ln\left(\frac{P}{\rho^\gamma}\right)$ the specific entropy

  - **buoyancy frequencies:** $N_i^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial i}\frac{\partial}{\partial i}\left(\ln\frac{P}{\rho^\gamma}\right)$ i={R,z}
  - in protoplanetary disks: $\Omega \approx \Omega_{\rm K}(R)$ $\quad |N_R^2| \sim h^2\Omega_{\rm K}^2 \quad |N_z^2| \ll |N_R^2|$

→ **[1]:** $\Omega_{\rm K}^2 + N_R^2 > 0$ always guaranteed, even when $N_R^2 < 0$

→ **[2]:** $N_z^2 > 0$ also verified, even where the disk surface is hot

- Protoplanetary disks should thus be linearly stable against ***adiabatic*** perturbations. But, should disk perturbations behave adiabatically?

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  → key role of **thermal diffusion timescale**

diffusion timescale over length scale H

opacity

$$\tau_{\rm diff}\Omega \approx 3 \times \left(\frac{\kappa}{1\,{\rm cm}^2\,{\rm g}^{-1}}\right)\left(\frac{h}{0.06}\right)^2\left(\frac{R}{20\,{\rm au}}\right)^{-3.5}$$

Nelson+ 13

→ **very short diffusion timescales at large R!**

Vertical shear instability (VSI)

- **Linear** instability in **3D axisymmetric** disks driven by **vertical shear** $(\partial\Omega/\partial z \neq 0)$ and **rapid thermal diffusion**

  Urpin & Branderburg 98, Nelson+ 13, Barker & Latter 15…

  - vertical shear due to **radial stratification** cf. **thermal wind** equation: $$R\frac{\partial\Omega^2}{\partial z} = -\mathbf{e}_\varphi \cdot \frac{\nabla\rho \times \nabla P}{\rho^2} = \frac{\partial T}{\partial R}\frac{\partial S}{\partial z} - \frac{\partial T}{\partial z}\frac{\partial S}{\partial R}$$

- Both conditions allow to violate **Solberg-Høiland's** second stability criterion

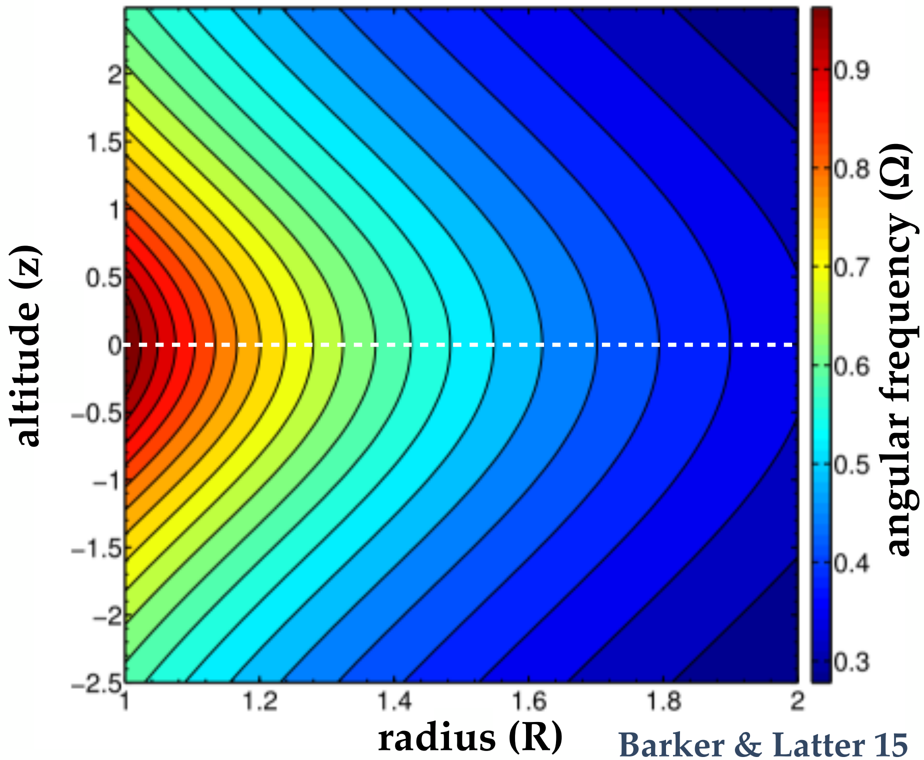
Péclet number

$$\mathbf{Pe}\, N_z^2 + \frac{2R}{\Omega_K}\frac{\partial\Omega}{\partial z}\frac{\partial P}{\partial z}\frac{\partial S}{\partial R} > 0$$

$\rightarrow$ 0 when thermal diffusion time $\rightarrow$ 0

< 0

- NB: can be further generalized by the addition of viscosity



Barker & Latter 15

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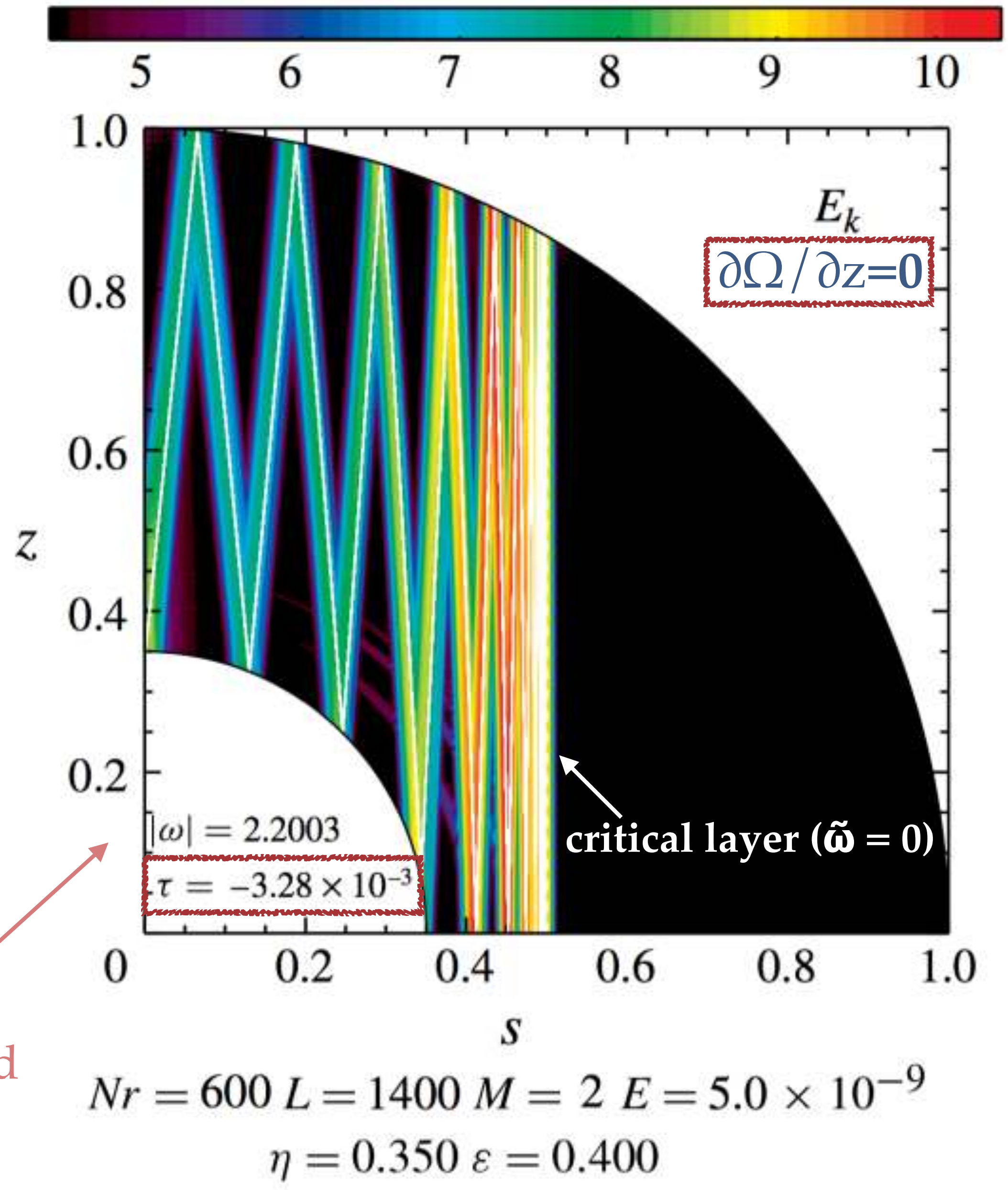
Baruteau & Rieutord 13 (J. Fluid Mech.)



(analogous to **IGWs** propagating towards a critical level, as we've heard in C. Staquet's and T. Rogers' talks)

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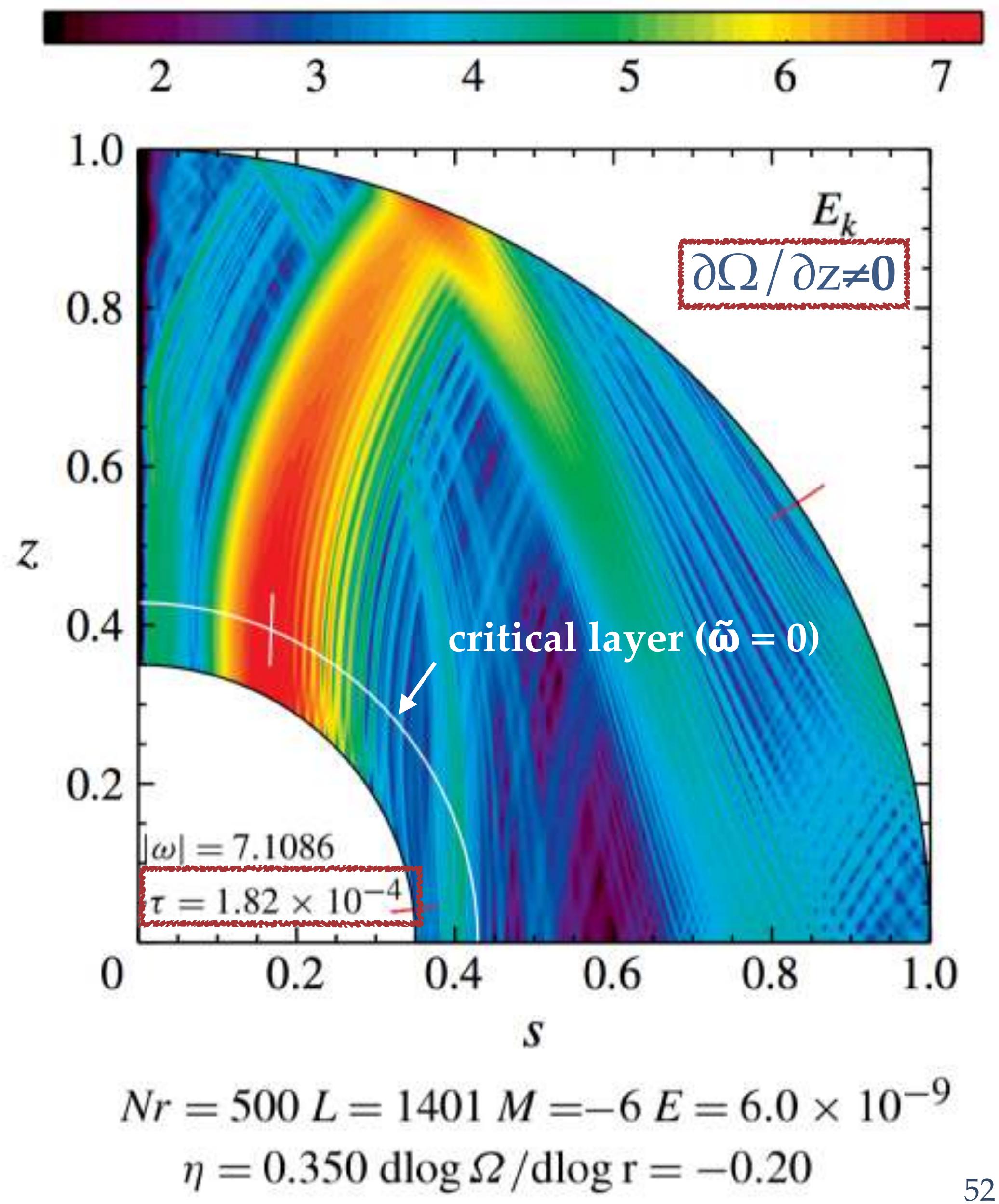

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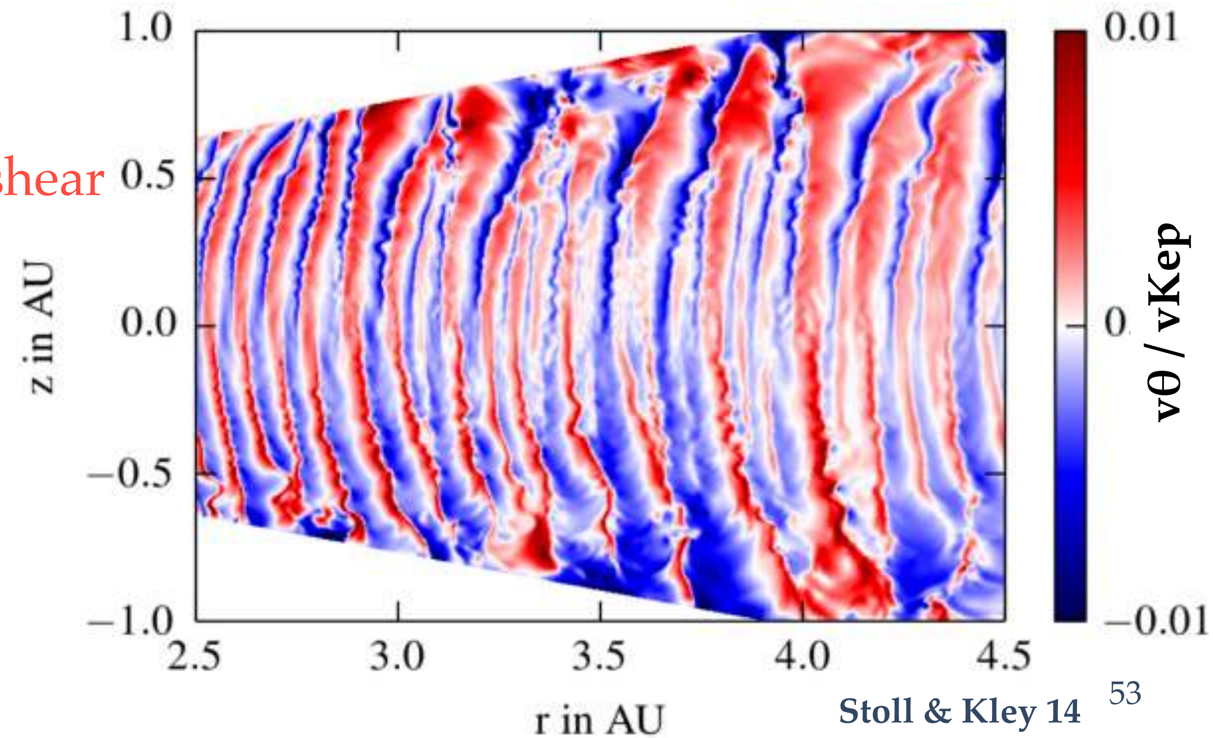

Stoll & Kley 14

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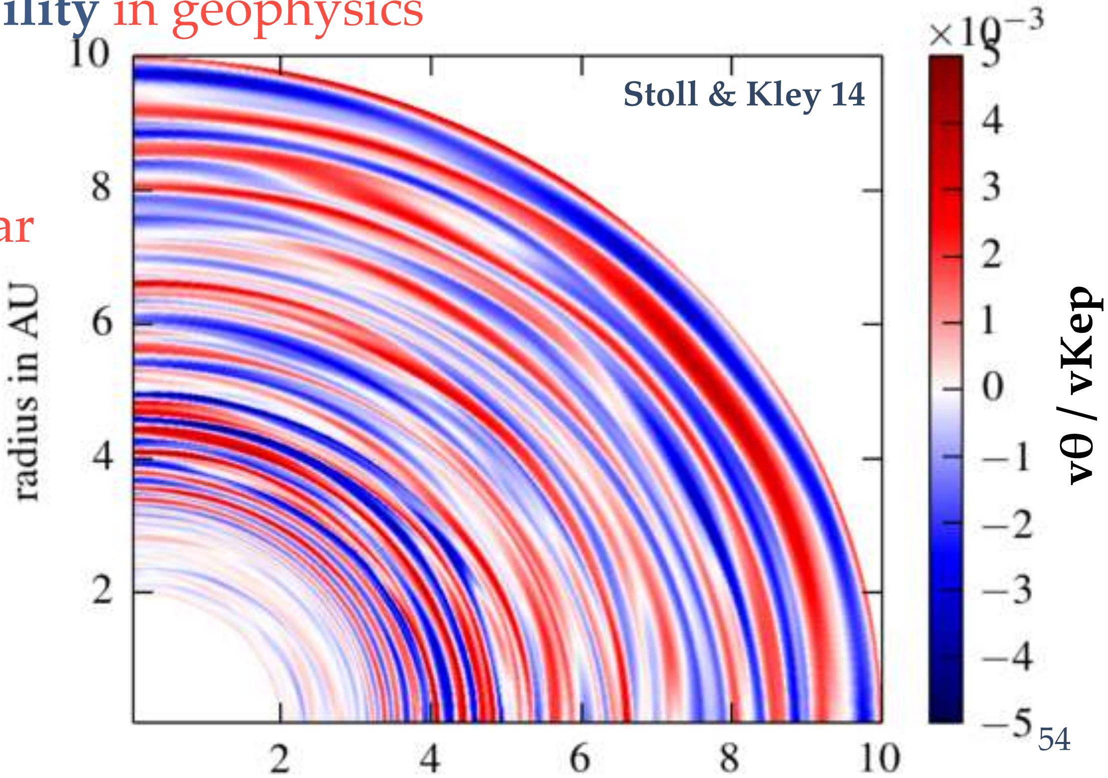
- **Moderate transport** of angular momentum: $\alpha \sim [10^{-6} - 10^{-4}]$

  Stoll & Kley 14, Richard+ 16



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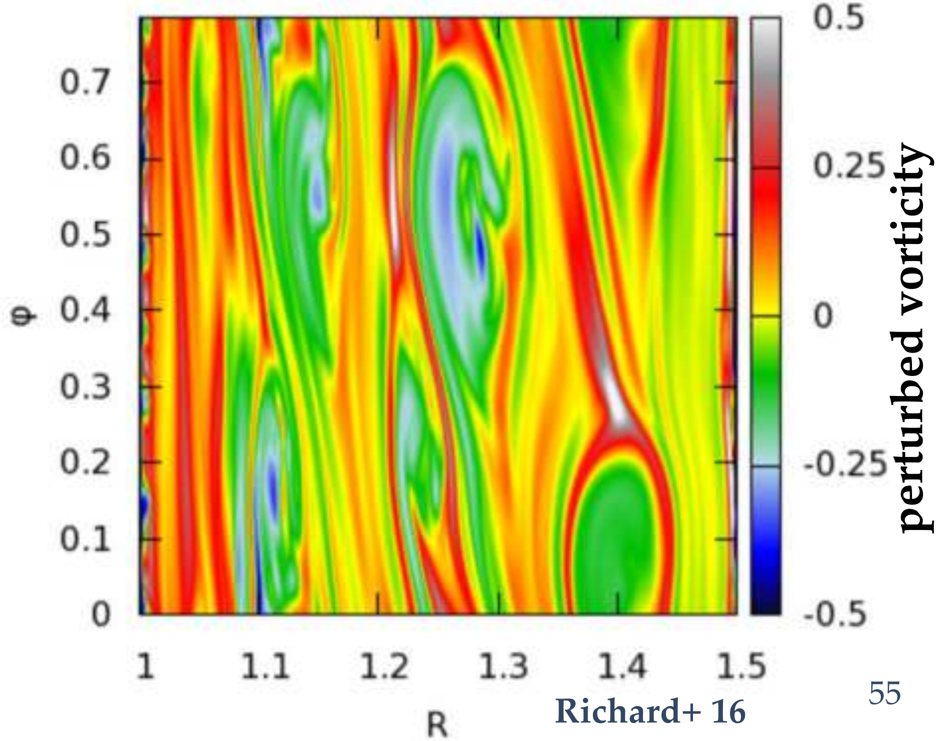
- Non-linear saturation into RWI-induced **vortices**?



Richard+ 16

Subcritical baroclinic instability (SBI)

- **Non-linear** instability driven by a **radially decreasing entropy** profile $(N_R^2 < 0)$ and **rapid thermal diffusion** in the presence of **non-axisymmetric** perturbations

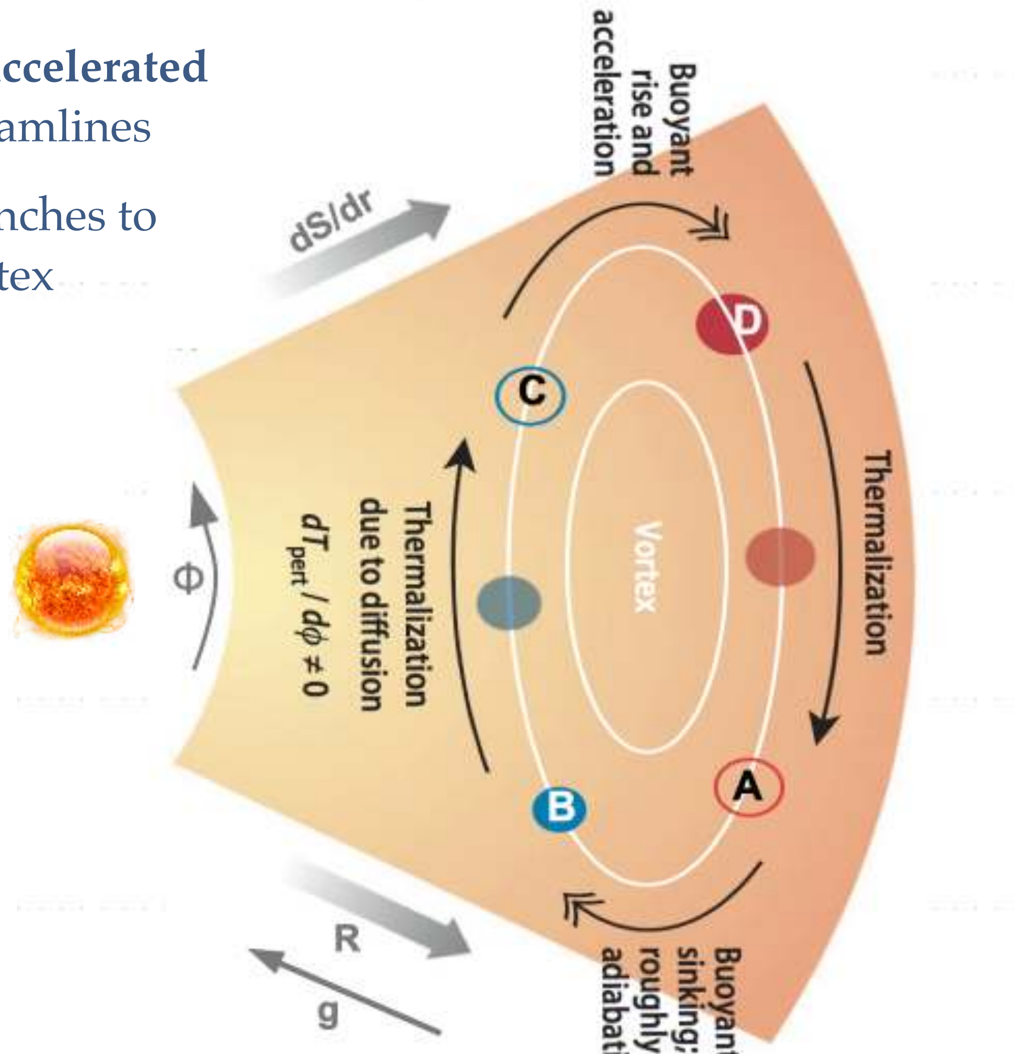
Klahr & Bodenheimer 03, Petersen+ 07, Lesur & Papaloizou 10, Barge+ 16…

- $N_R^2 < 0$ for fluid particles to be **buoyantly accelerated** along the radial branches of the vortex's streamlines
- rapid thermal diffusion along azimuthal branches to **maintain** radial **entropy gradient** across vortex

$\rightarrow$ vortex **strengthens**

$$\frac{D\omega_z}{Dt} = \boxed{\frac{(\nabla\rho \times \nabla \mathbf{T}) \cdot \hat{\mathbf{z}}}{\rho}}$$

"baroclinic term"



Armitage 11

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Petersen+ 07

- Can both criteria for SBI be met **simultaneously** in disks?
- SBI vortices excite **density waves** that **extract** angular momentum away from vortex ($\alpha \lesssim 10^{-3}$), vortices thus **move** radially inwards!

Paardekooper+ 10

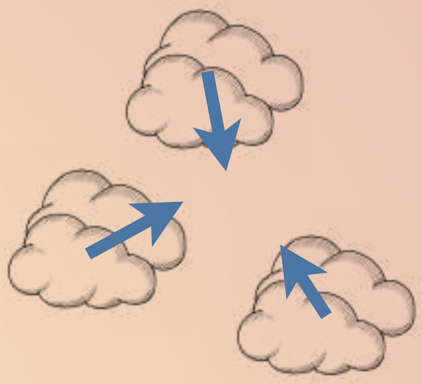
- Long-term evolution of SBI vortices?

Convective over stability (COS)

- **Linear** sibling of the SBI in **3D axisymmetric** disks — **Klahr & Hubbard 14, Lyra 14, Latter 16**
    - fluid particles now follow **horizontal epicyclic motions** instead of vortex streamlines
- growth rate of instability: $$\omega_i \approx -\Omega \left(\frac{N_R}{\Omega}\right)^2 \frac{\Omega\tau_c}{1+\Omega^2\tau_c^2}$$
    - hence the condition $N_R^2 < 0$ for **growth**, which is **fastest** for $\Omega\tau_c \sim 1$
- Connection to Vertical Shear Instability?
- Non-linear saturation into SBI vortices?

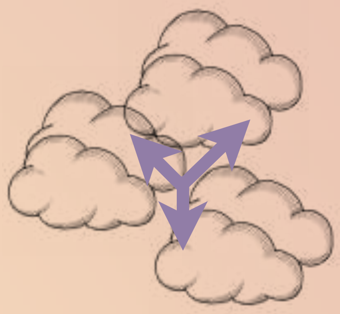
Gravitational instability (GI)

- **Linear** instability resulting from the competition between gas **self-gravity**



Gravitational instability (GI)

- **Linear** instability resulting from the competition between gas **self-gravity**, **pressure forces**



Gravitational instability (GI)

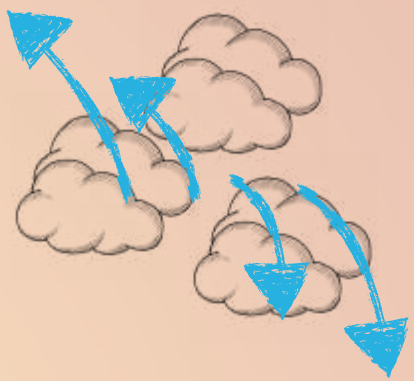
- **Linear** instability resulting from the competition between gas **self-gravity**, **pressure forces** and **differential rotation**

dispersion relation of the linearised governing equations:

$$\underbrace{(\omega - m\Omega)^2}_{=\,\tilde{\omega}^2(k)} = \kappa^2 - 2\pi G\Sigma|k| + k^2 c_s^2$$

with ω the wave frequency
m the azimuthal wavenumber
k the radial wave-vector
κ the radial epicyclic frequency:

$$\kappa^2 = \frac{1}{R^3}\frac{dj^2}{dR} \equiv \Omega^2$$



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$$\underbrace{(\omega - m\Omega)^2}_{= \tilde{\omega}^2(k)} = \underbrace{\kappa^2}_{= \Omega^2} - 2\pi G\Sigma|k| + k^2 c_s^2 \quad \text{is minimum for} \quad |k| = \frac{\pi G\Sigma}{c_s^2}$$

that minimum is equal to $\Omega^2 \times \dfrac{Q^2 - 1}{Q^2}$ with $\boxed{Q = \dfrac{c_s\Omega}{\pi G\Sigma}}$ the Toomre parameter

↑ **< 0 for Q < 1**

→ **criterion for gravitational instability: Q < 1** **Toomre 64**

Gravitational instability (GI)

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$$Q = \frac{c_s \Omega}{\pi G \Sigma} < 1$$

Toomre 64

- Instability **criterion** more likely met in the **early** ($\lesssim 10^5$ yr) evolution of **massive** ($M_{disk} \gtrsim 0.1\ M_\star$) disks, typically at $R \gtrsim$ 30-50 au from the star

- Non-linear evolution depends on the disk's **cooling** timescale — Gammie 01

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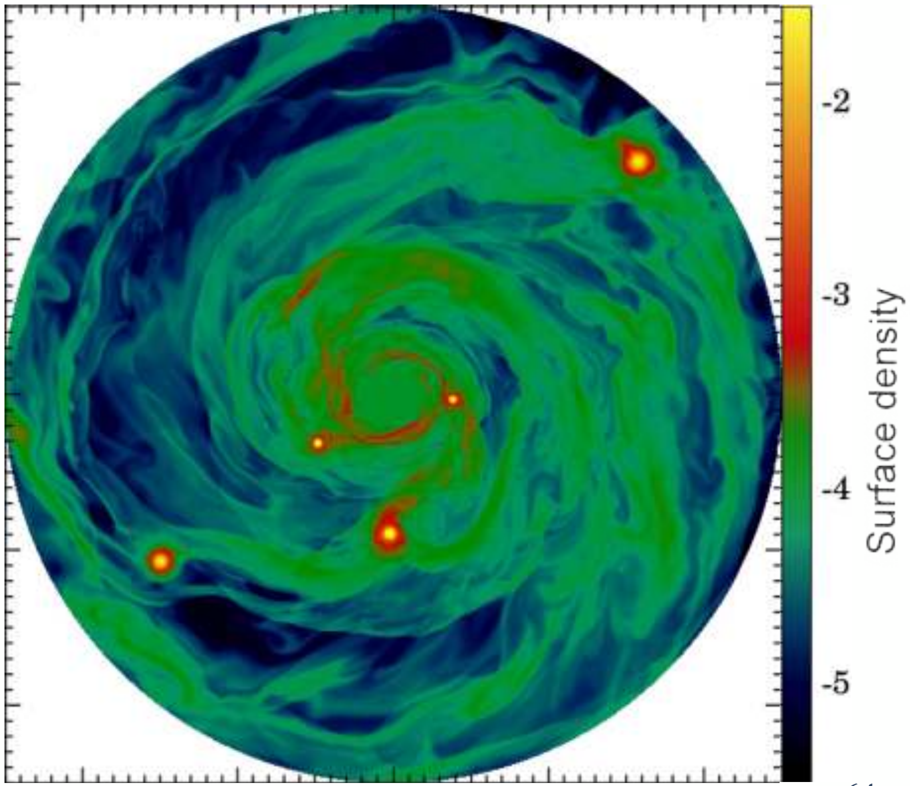
1) $\tau_{cool}\Omega \lesssim 3-5$

→ the disk **fragments** and breaks up into bound **clumps** with typical mass $\gtrsim$ **$M_{Jupiter}$**



Paardekooper+ 11

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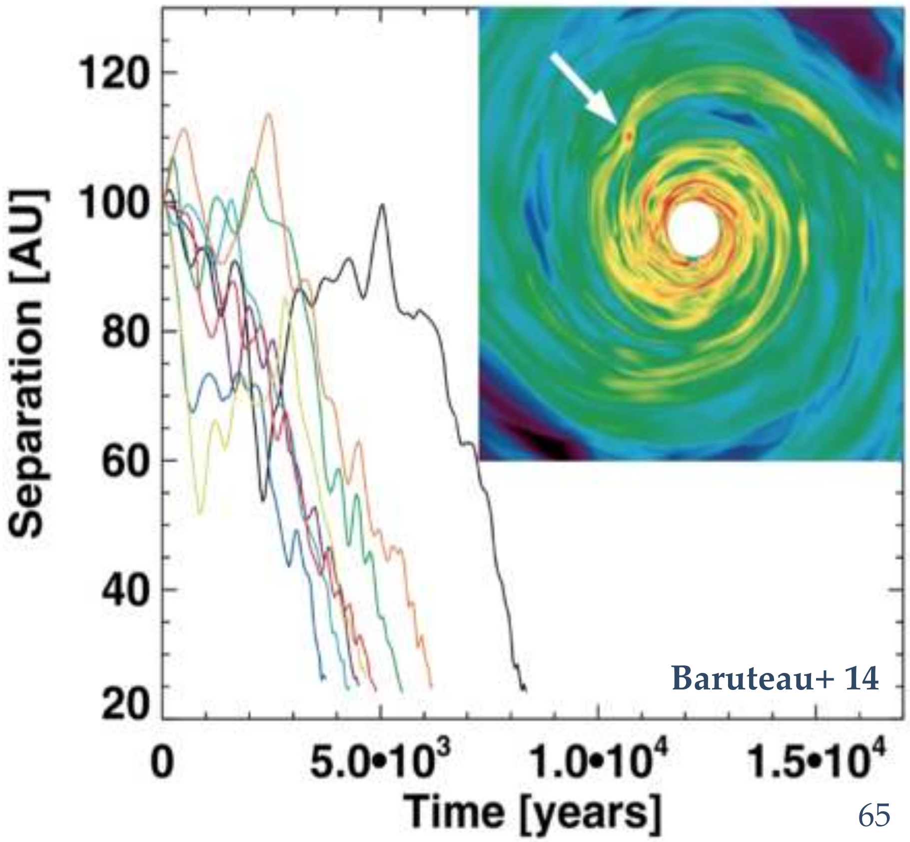
→ internal and orbital **evolutions**?



Baruteau+ 14

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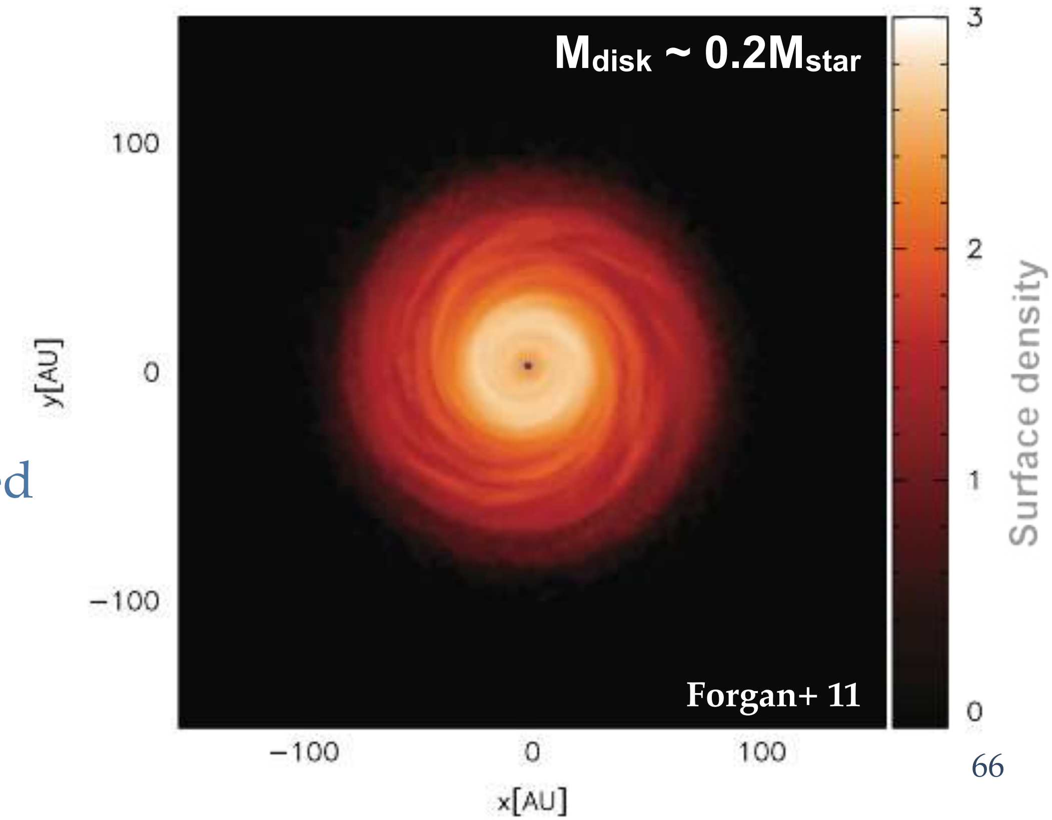
2) $\tau_{cool}\Omega \gtrsim 3-5$

→ the disk reaches a quasi steady-state with **turbulent** mass **accretion** mediated by spiral **waves**



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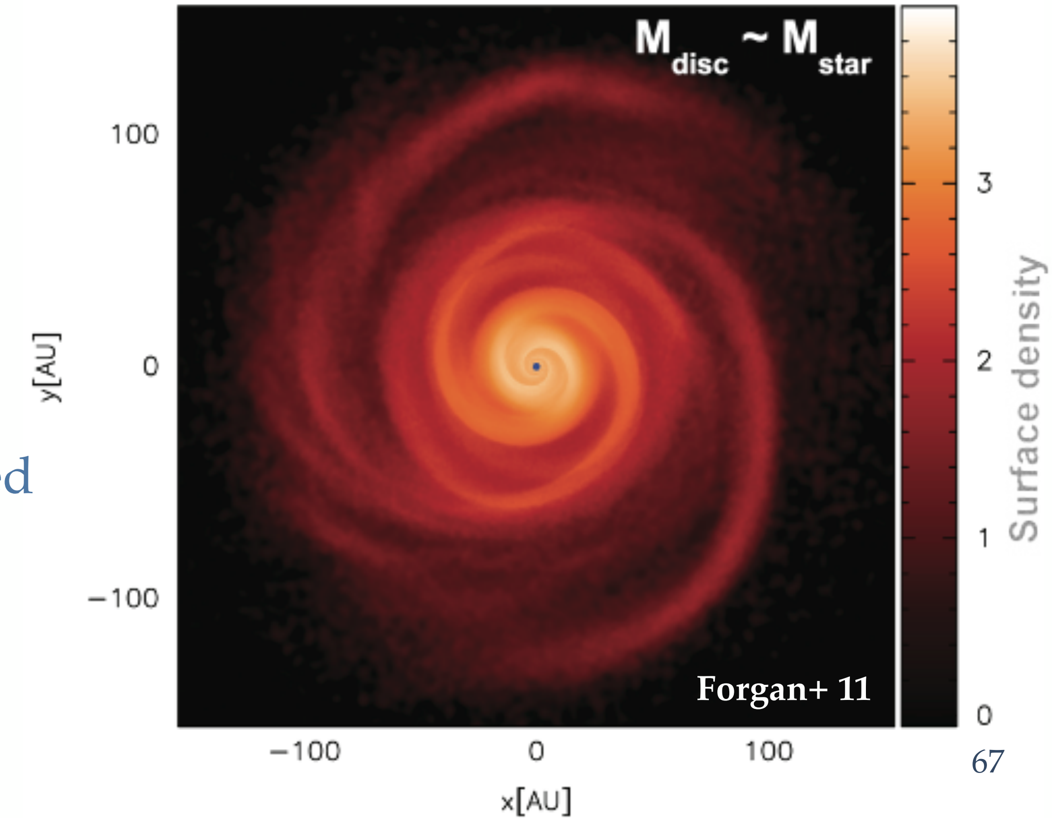
$$2)\ \tau_{cool}\Omega \gtrsim 3-5$$

→ the disk reaches a quasi steady-state with **turbulent** mass **accretion** mediated by spiral **waves**… at a rate possibly ≠ from the prediction of an "alpha" disk model (depends on disk mass)

Balbus & Papaloizou 99, Cossins+ 09…



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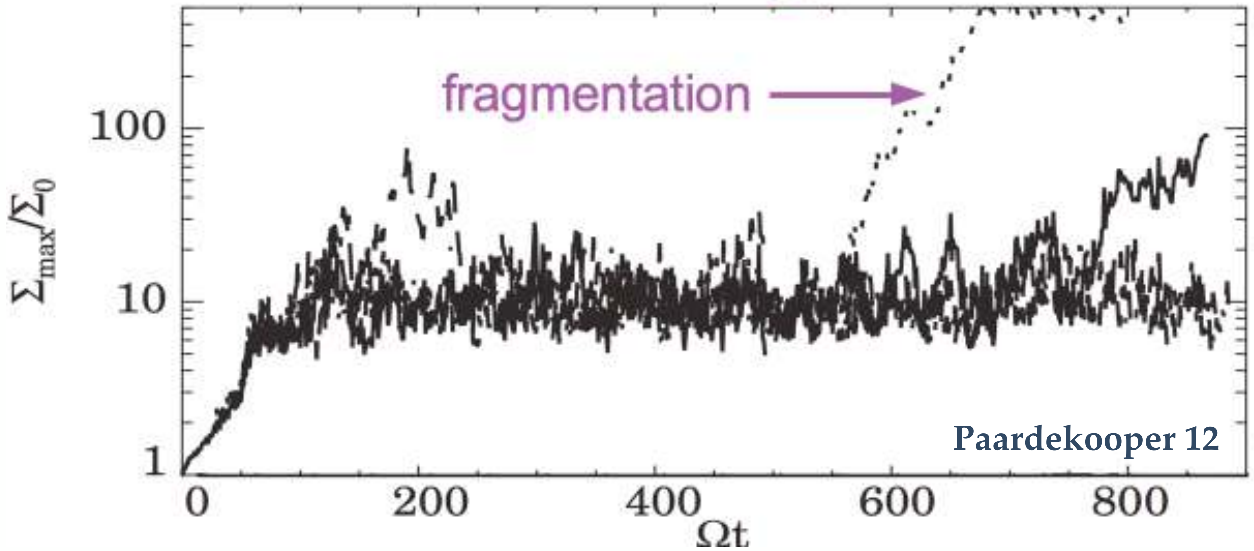
- Is fragmentation **stochastic**?

$\tau_{cool}\Omega = 7$

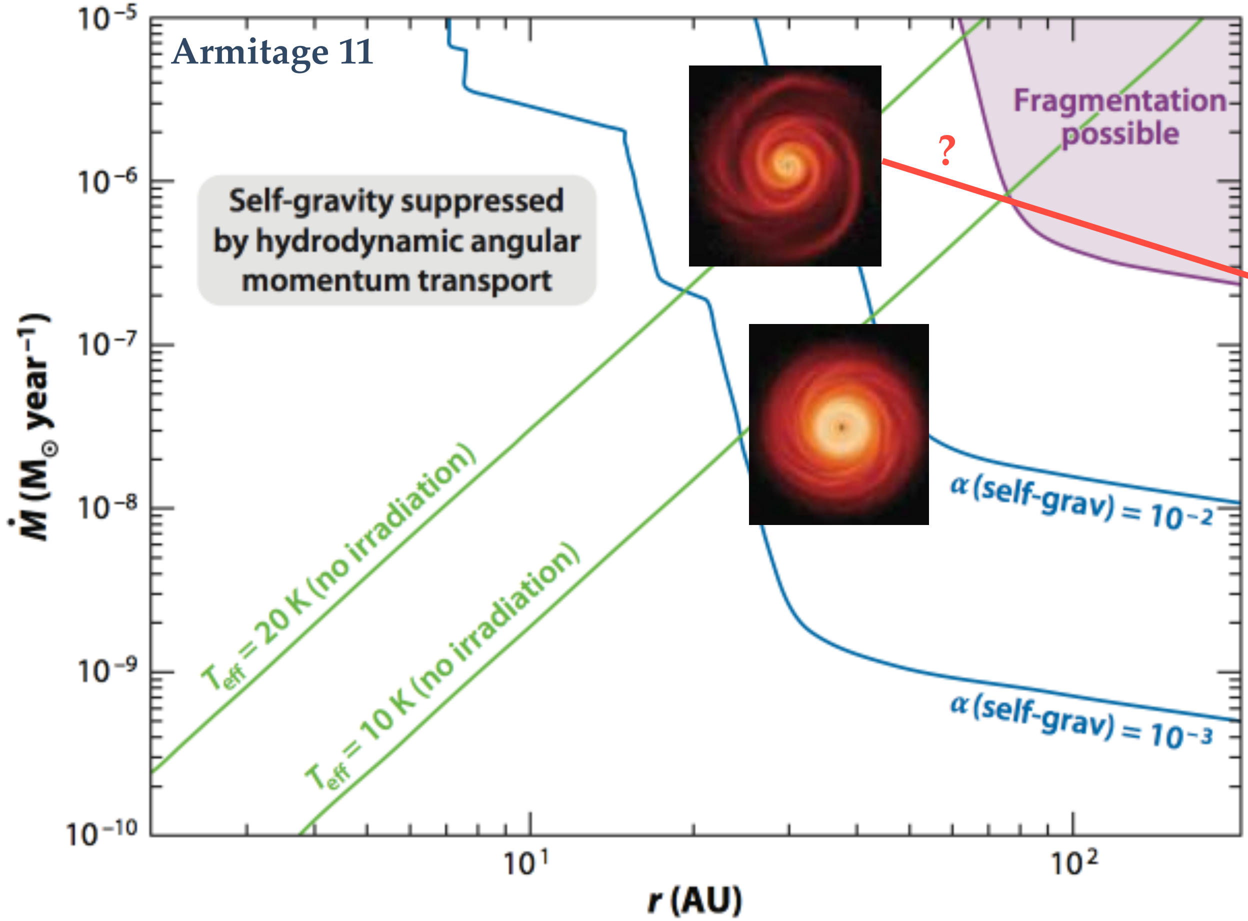


Paardekooper 12

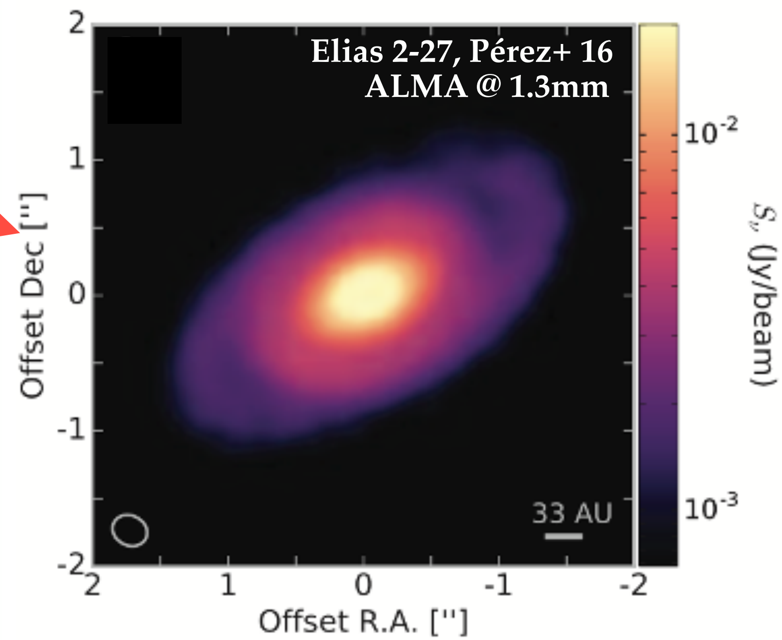
Gravitational instability (GI): take away

- GI can be active in the **outer** (>30 AU) parts of **young**, still **massive** disks

  → outcome still uncertain: does GI always lead to fragmentation?



alpha viscous model of self-gravitating disk in thermal equilibrium



Elias 2-27: a ~1 Myr ~0.5 $M_\odot$ star surrounded by a ~0.1 $M_\odot$ disk

→ gravitational instability?

Hydrodynamical instabilities: take away

name	linear?	driver	non-linear evolution
Gravitational instability	✓	$Q \lesssim 1$	density waves fragmentation
Rossby wave instability	✓	extremum of $\mathcal{L} = \frac{\Sigma}{2\omega_z}\left(\frac{P}{\Sigma^\gamma}\right)^{2/\gamma}$	V O R T I C E S
Subcritical baroclinic instability	✗	$\nabla_r S < 0$	
Convective over stability	✓	$\nabla_r S < 0$	
Vertical shear instability	✓	$\nabla_r S < 0$ or $\nabla_r T < 0$	

and short thermal diffusion / cooling timescale for all

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! there may be **other** hydrodynamical instabilities in disks !

see, e.g., reviews by Fromang & Lesur 17, Lyra & Umurhan 19…