Effect of Background Mean Flow on PSI of Internal Wave Beams

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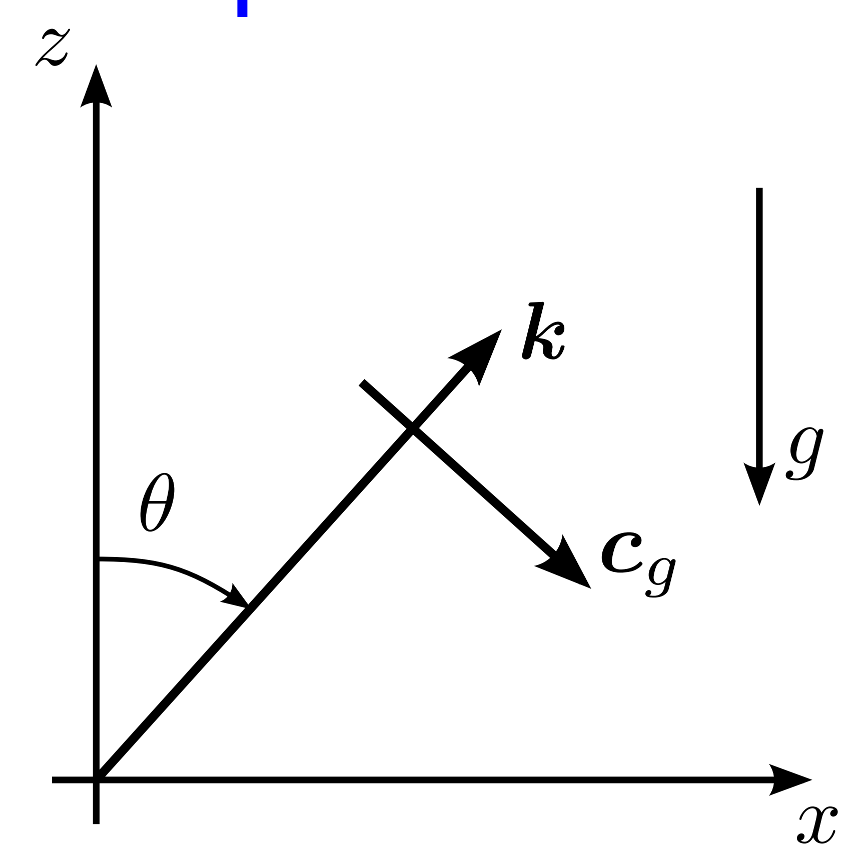
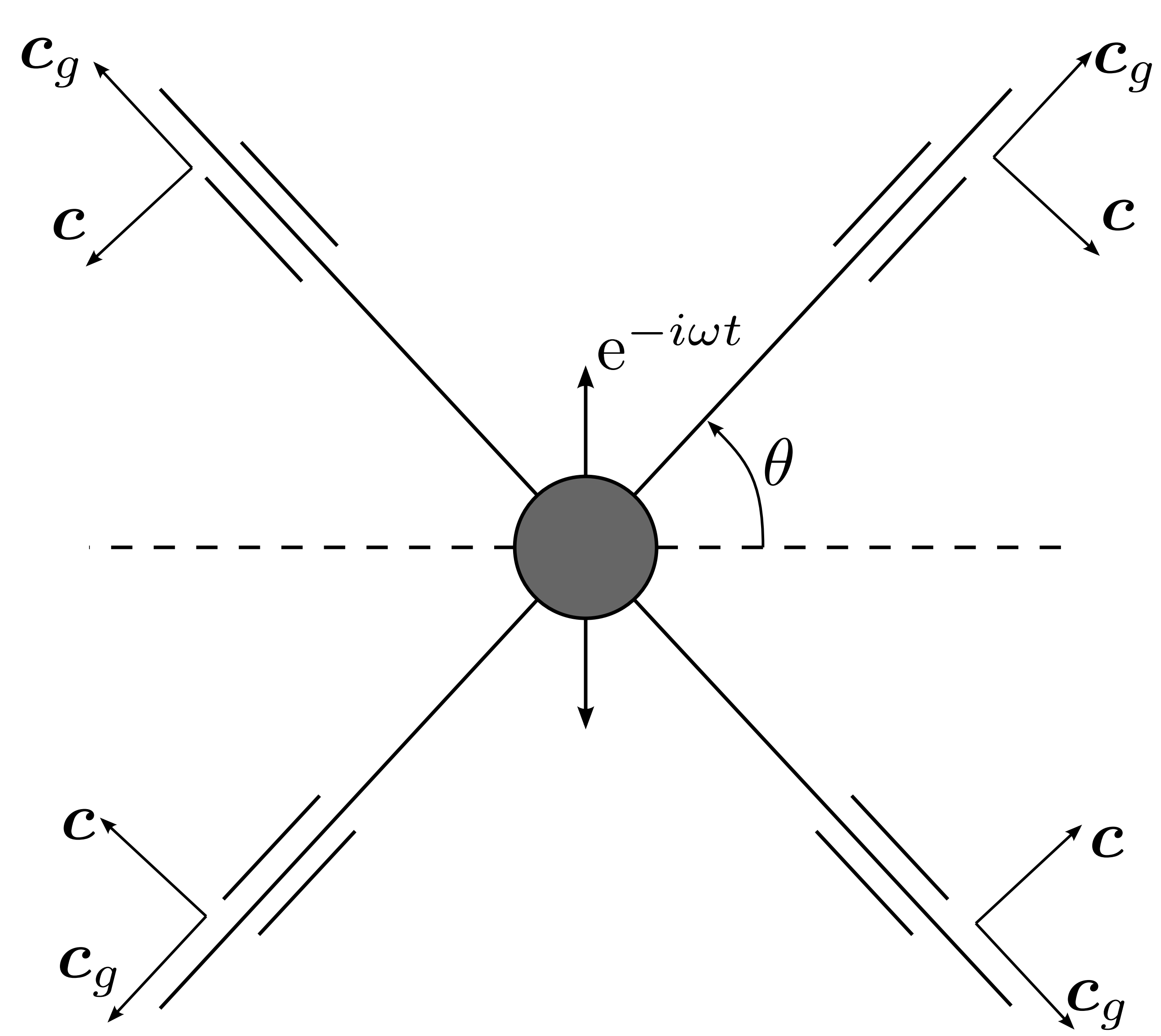
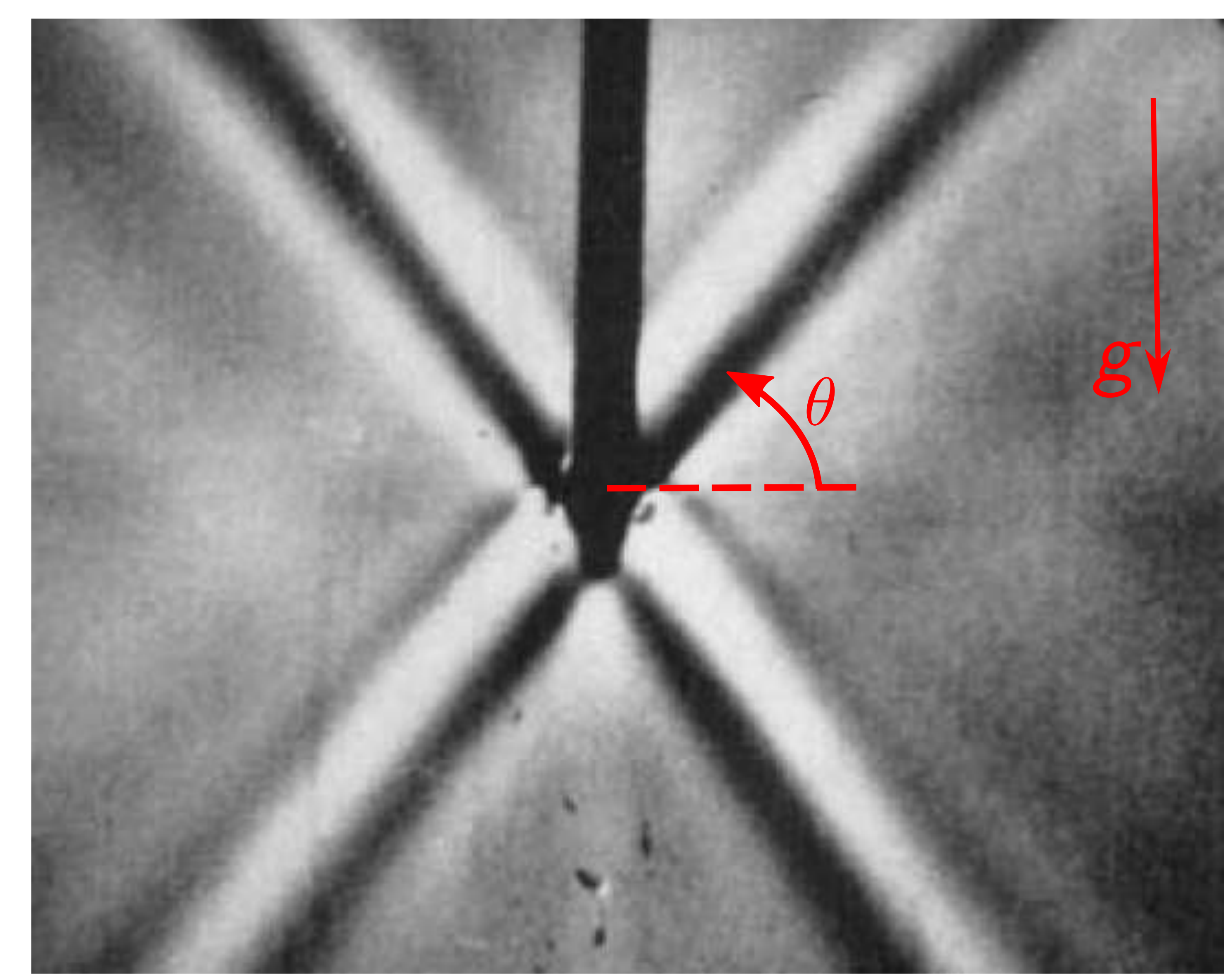
Background

- Gravity waves in stratified fluids are anisotropic
  - $N^2 = -\frac{g}{\overline{\rho}}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z}$
  - $\omega = N\sin\theta$ (independent of k)
  - $\boldsymbol{c_g} = \boldsymbol{\nabla_k}\omega$ is perpendicular to $\boldsymbol{c} = \frac{\omega}{k^2}\boldsymbol{k}$



- Internal wave beams (IWB) are manifestations of this anisotropy





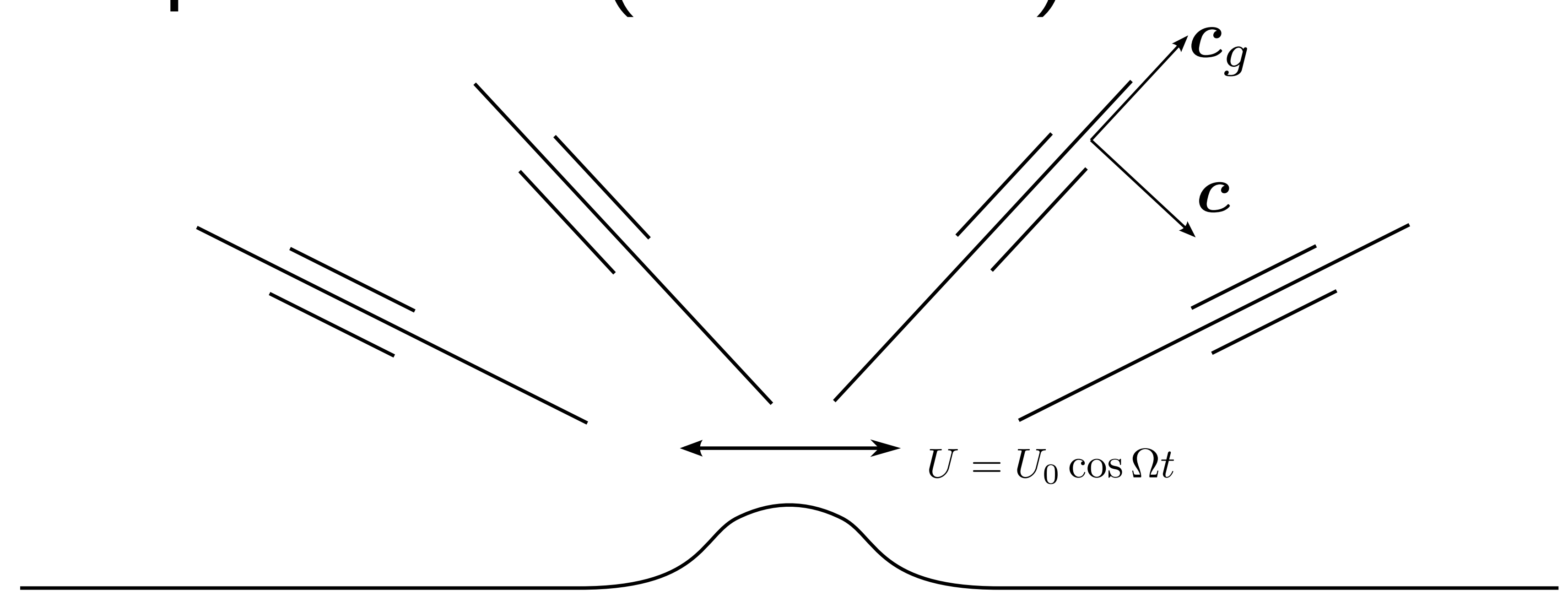
Mowbray & Rarity (1967)

- 2D infinitely long, uniform IWB are exact nonlinear solutions in an inviscid, uniformly stratified Boussinesq fluid

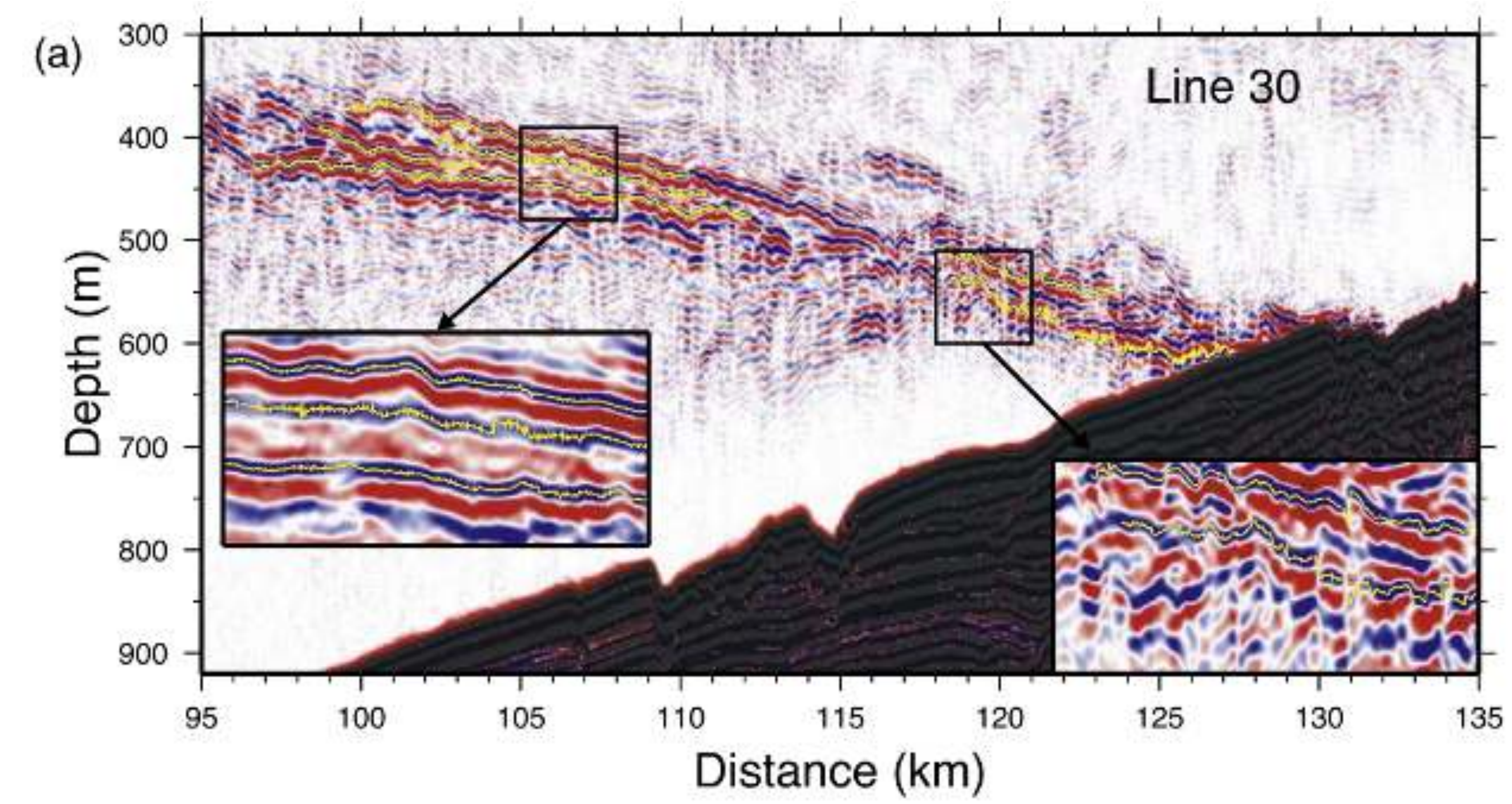
Background

- Internal wave beams play an important part in the interaction of the barotropic tide with sea-floor topography (tidal conversion)

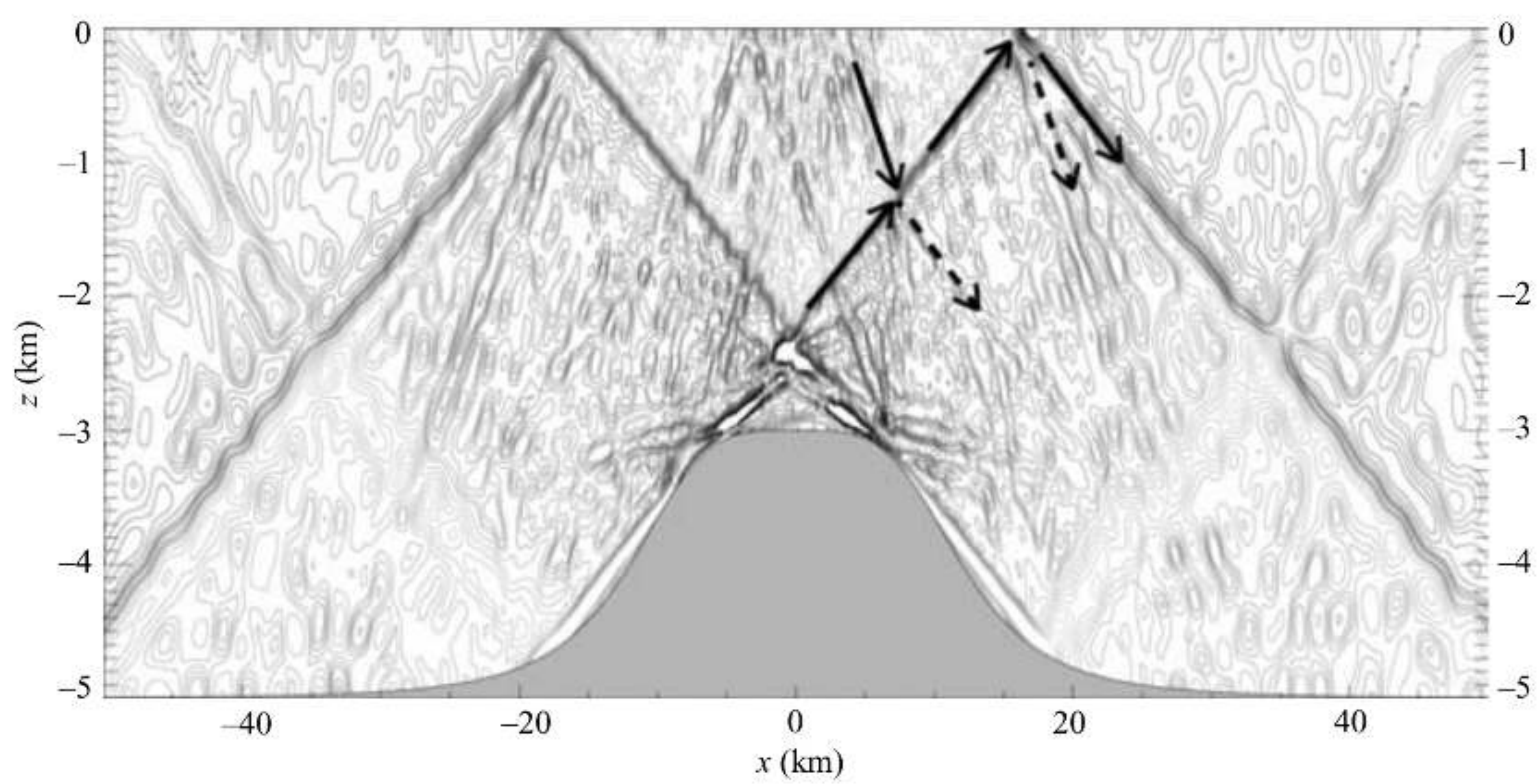
  - Simple model (Bell 1975)



  - Field observation (Holbrook 2005)

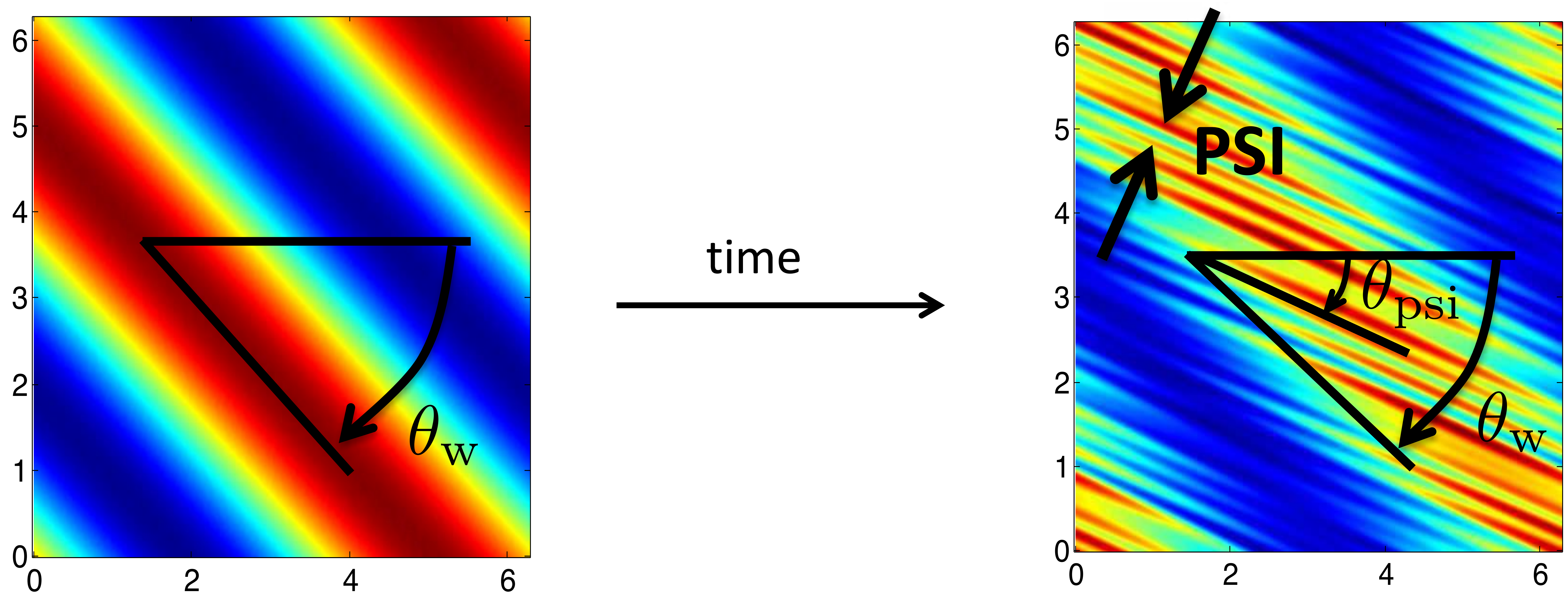


  - Fully numerical simulation of oscillatory flow over a ridge of finite steepness (Lamb 2003)



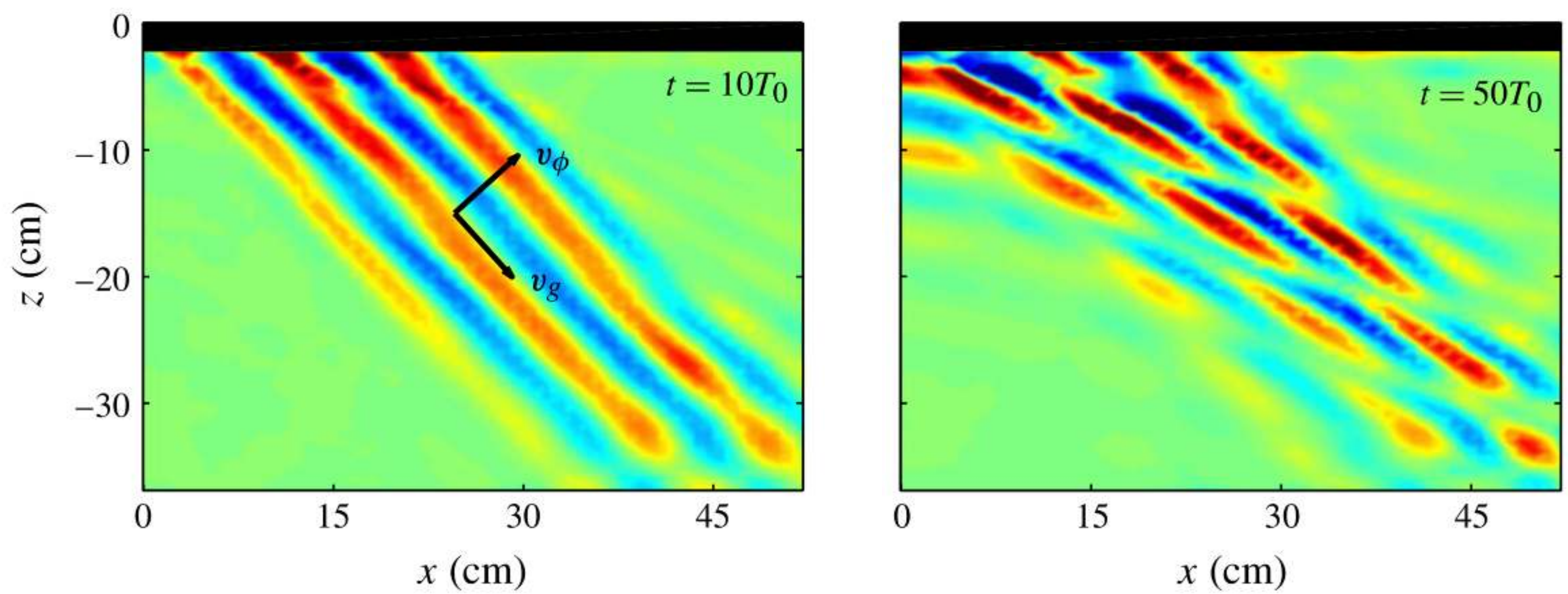
Stability of monochromatic internal waves

- Monochromatic internal waves of infinite extent are always unstable in the inviscid limit; Mied (1976)
- Dominant mode of instability, known as *Parametric Subharmonic Instability* (PSI); McEwan (1977), Drazin (1977)
    - Fine-scale disturbances with sub-harmonic frequencies, $\omega/2$, due to resonant triad mechanism
- Analytical, numerical, and experimental investigations suggest subsequent breakdown leads to turbulence and mixing (1990's—2000's)



Stability of internal wave beams

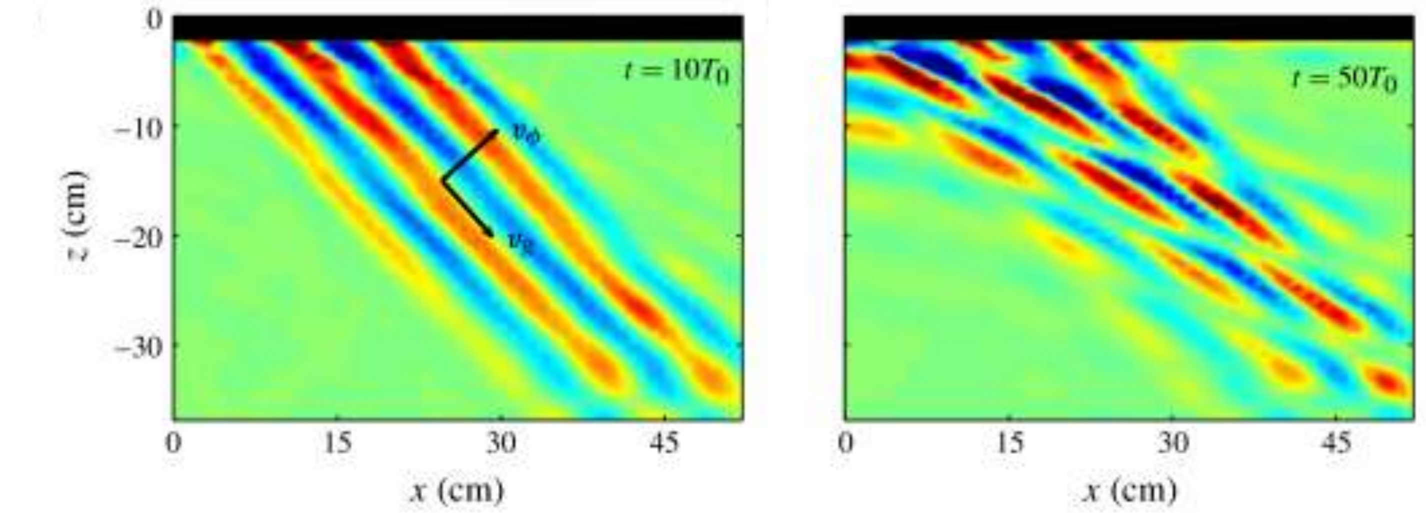
- Internal waves in nature and laboratory are not always monochromatic
- **Finite width** of internal wave beam plays crucial role in instability (Bourget *et al.* 2014, Karimi & Akylas 2014)



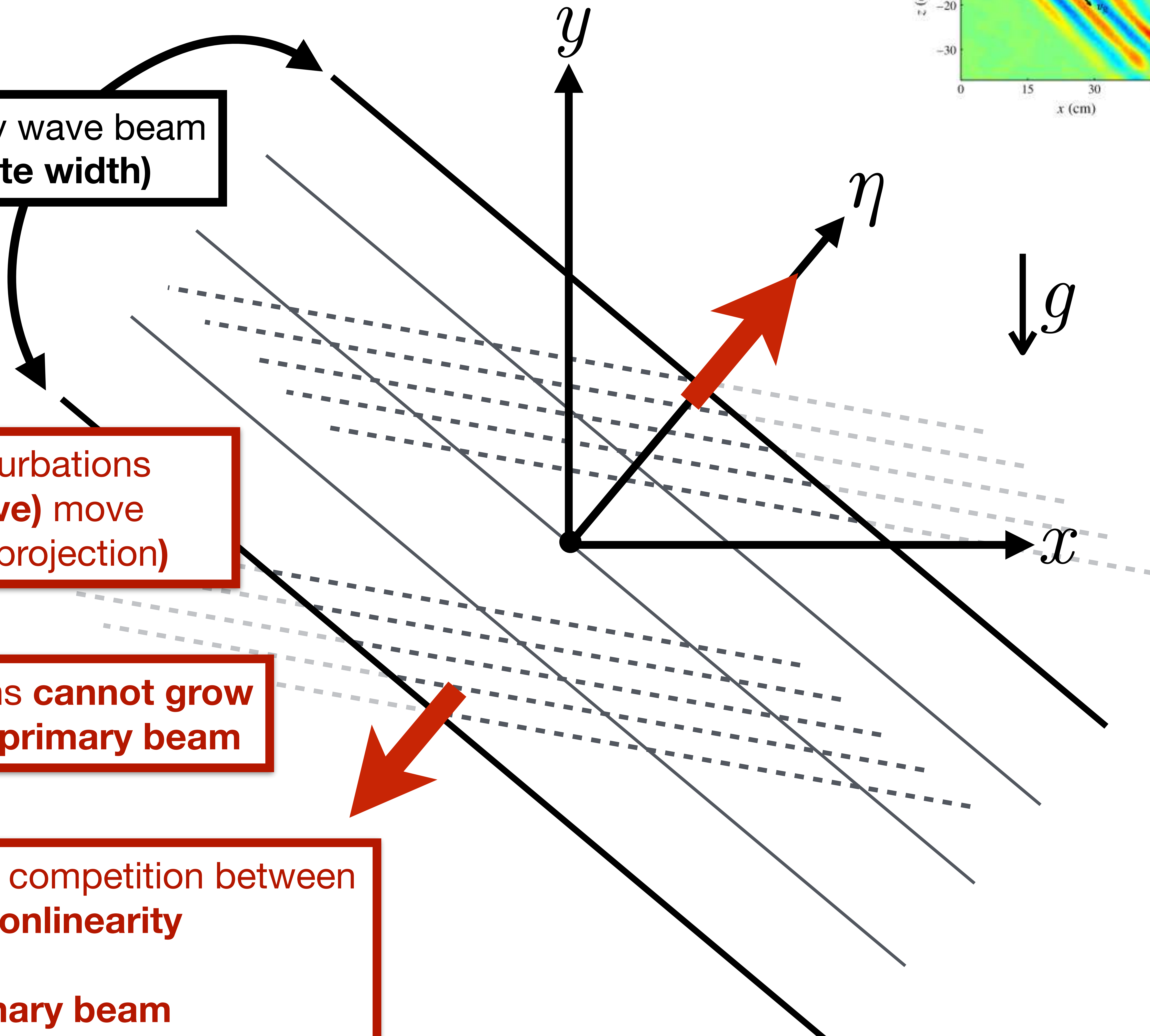
Bourget et al. (2013)

- **Weakly nonlinear model** (Karimi & Akylas 2014, 2017) for locally confined waves shows:
  - Without background rotation, profile must be **nearly monochromatic** for PSI to exist
  - With background rotation, profile can be general but **primary wave frequency ~ 2*f***

Stability of internal wave beams



Bourget et al. (2013)



Primary wave beam
(finite width)

Fine-scale subharmonic perturbations **(modulated by primary wave)** move according to **group velocity** (projection)

Perturbations **cannot grow outside of primary beam**

For PSI (perturbation growth), competition between
time to grow via nonlinearity
vs.
time to leave primary beam

Karimi et al. (2014)

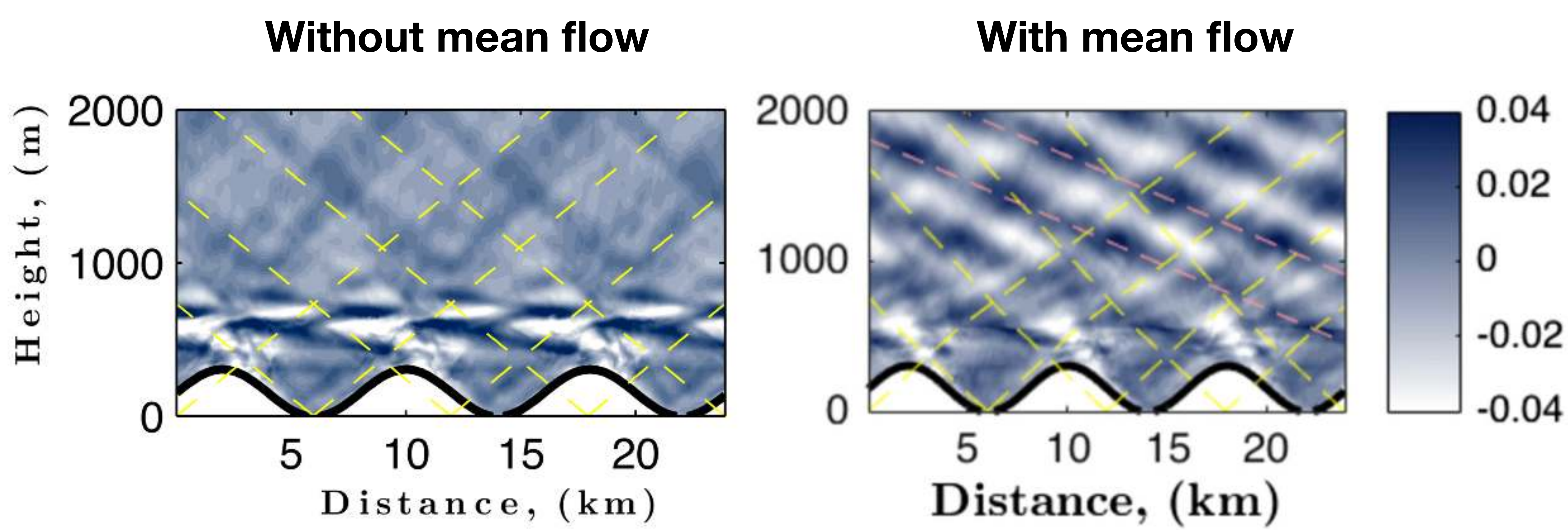
PSI in the natural environment – locally confined + mean flows

Theory/simulation **overpredicts severity of PSI** compared to observations around the critical latitude (Alford *et al.* 2007, MacKinnon *et al.* 2013)

Vast body of work on plane waves, but only recently **locally confined beams** (Bourget *et al.* 2014, Karimi *et al.* 2014)

+

Large-scale currents ubiquitous, but mostly ignored



Richet et al. (2017)

How do **background mean flows** affect internal wave beam PSI?

PSI with mean flow — plane waves

Classical PSI — resonant interaction between primary wave and **fine-scale** subharmonics:

$$\omega_+ + \omega_- = \omega_0 \qquad \omega_0 \quad \longleftrightarrow \quad \omega_\pm = \frac{\omega_0}{2}$$

$$\boldsymbol{k}_+ + \boldsymbol{k}_- = \boldsymbol{k}_0$$

With mean flow, Doppler shifted dispersion relation:

$$\underbrace{(\omega - \overline{u}|\boldsymbol{k}|\sin\theta)^2}_{\substack{\equiv\sigma,\ \text{frequency} \\ \text{in fluid frame}}} \;=\; \sin^2\theta$$

Via resonant interaction theory, **in the fluid frame, PSI resonance is the same**: $\sigma_0 \quad \longleftrightarrow \quad \sigma_\pm = \frac{\sigma_0}{2}$

Mean flow only has **minor effect** on PSI for **plane waves**

- most unstable perturbations are still **short-scale**
- with rotation, mean flow changes cutoff frequency for propagating waves (i.e. Richet et al. 2017)

What about for **locally confined beams**?

Effect of mean flow on primary wave

- Primary wave with carrier wavevector

$$\boldsymbol{k}_0 = \hat{\boldsymbol{e}}_\eta$$

- Source frequency and angle of inclination are related by

$$\omega_0 = (1 + \overline{u}) \sin\theta$$

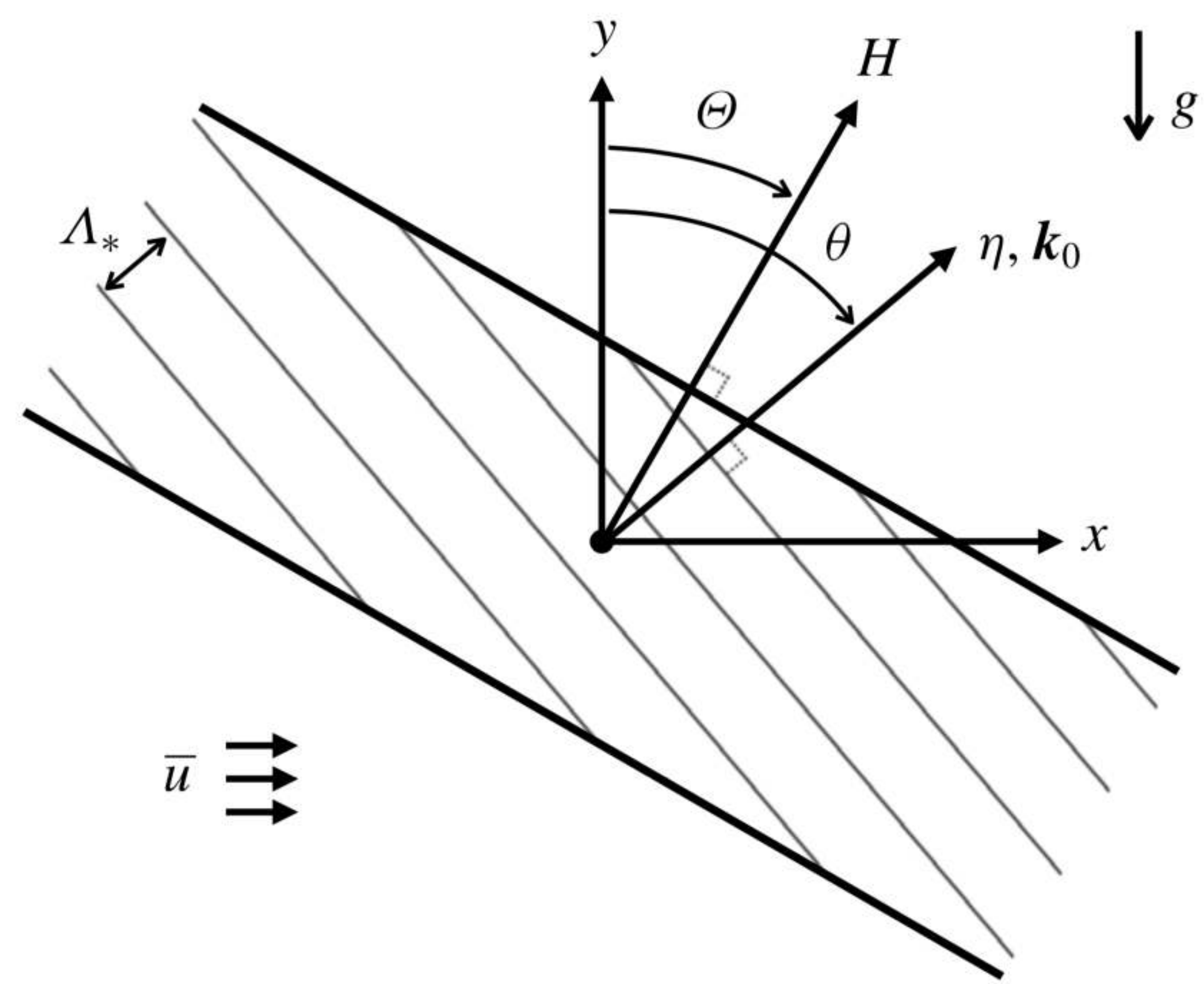
- Since locally confined profile, sidebands will propagate at different angle

$$H = \mu(x \sin\Theta + y \cos\Theta)$$

$$\tan\Theta = \frac{-c_{g,y}}{c_{g,x} + \overline{u}} = \frac{\sin\theta}{\cos\theta + \overline{u}/\cos\theta}$$

- Therefore, primary wave profile has form

$$\psi_0 = \rho_0 = \epsilon\Big\{Q(H)\mathrm{e}^{\mathrm{i}(\eta - \omega_0 t)} + \mathrm{c.c.}\Big\} + \ldots$$



Evolution of subharmonics with mean flow

- Distinguished limit where effects of **nonlinearity**, **group velocity**, **viscosity**, and **mean flow** balance

$$\psi_0 = O(\epsilon), \quad \boldsymbol{k}_\pm = O(\epsilon^{-1/2}), \quad \mu = \epsilon^{1/2}, \quad \overline{u} \to \epsilon^{1/2}\overline{u}, \quad \nu = 2\alpha\epsilon$$

- Must assume **small mean flow**. For *O(1)* mean flow, no instability! (see Fan & Akylas 2019)

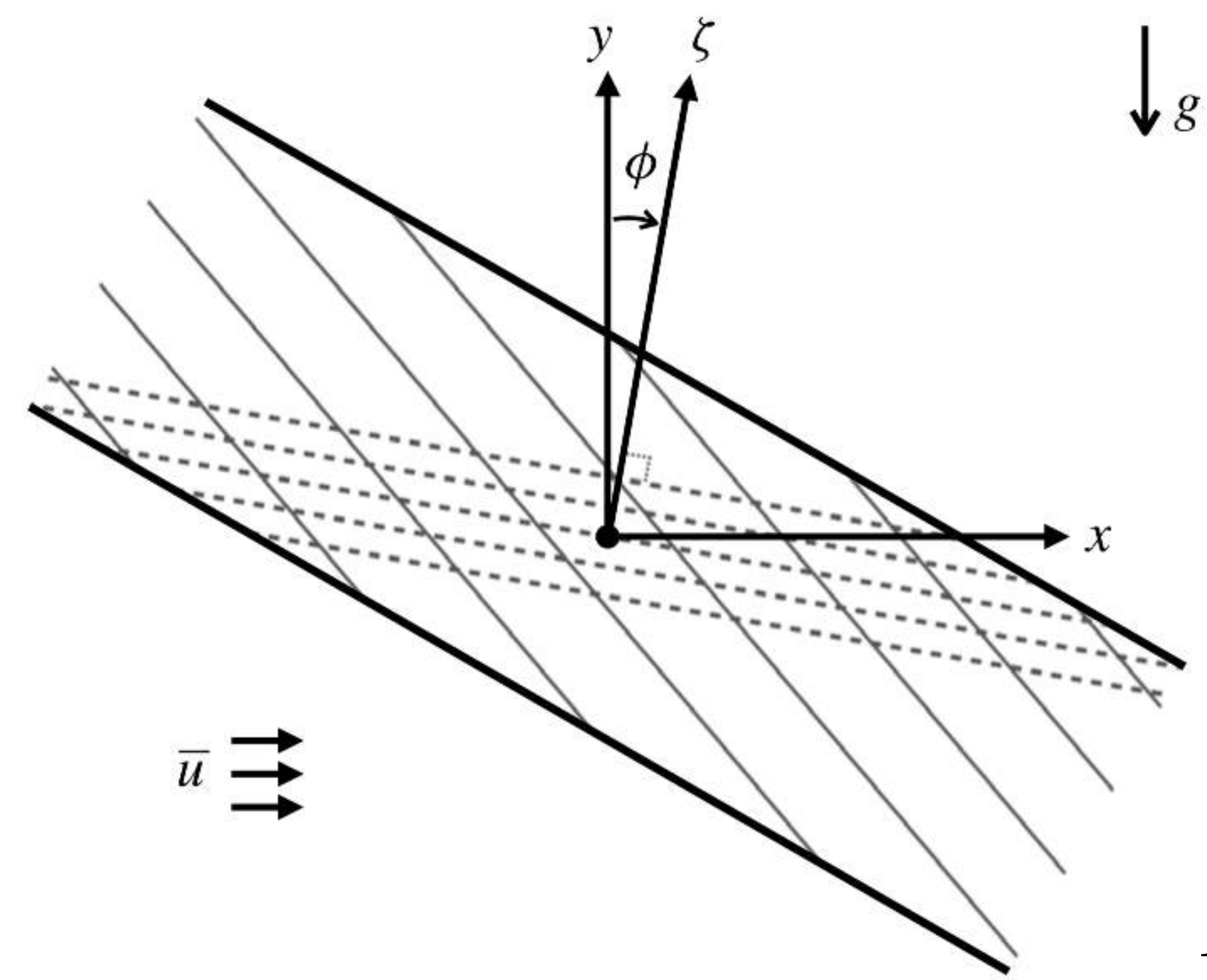
- Primary wave

$$\boldsymbol{k}_0 = \boldsymbol{e}_\eta + \ldots$$
$$\omega_0 = \sin\theta(1 + \epsilon^{1/2}\overline{u}) + \ldots$$

perturbation wavenumber

- Assume **fine-scale** perturbations

$$\boldsymbol{k}_\pm = \pm\frac{\kappa}{\epsilon^{1/2}}\boldsymbol{e}_\zeta + \frac{1}{2}\boldsymbol{e}_\eta + \ldots$$
$$\omega_\pm = \sin\phi\,(1 \pm \overline{u}\kappa) + \ldots$$



- To satisfy triad resonance conditions:

$$\sin\phi = \frac{\sin\theta}{2}$$

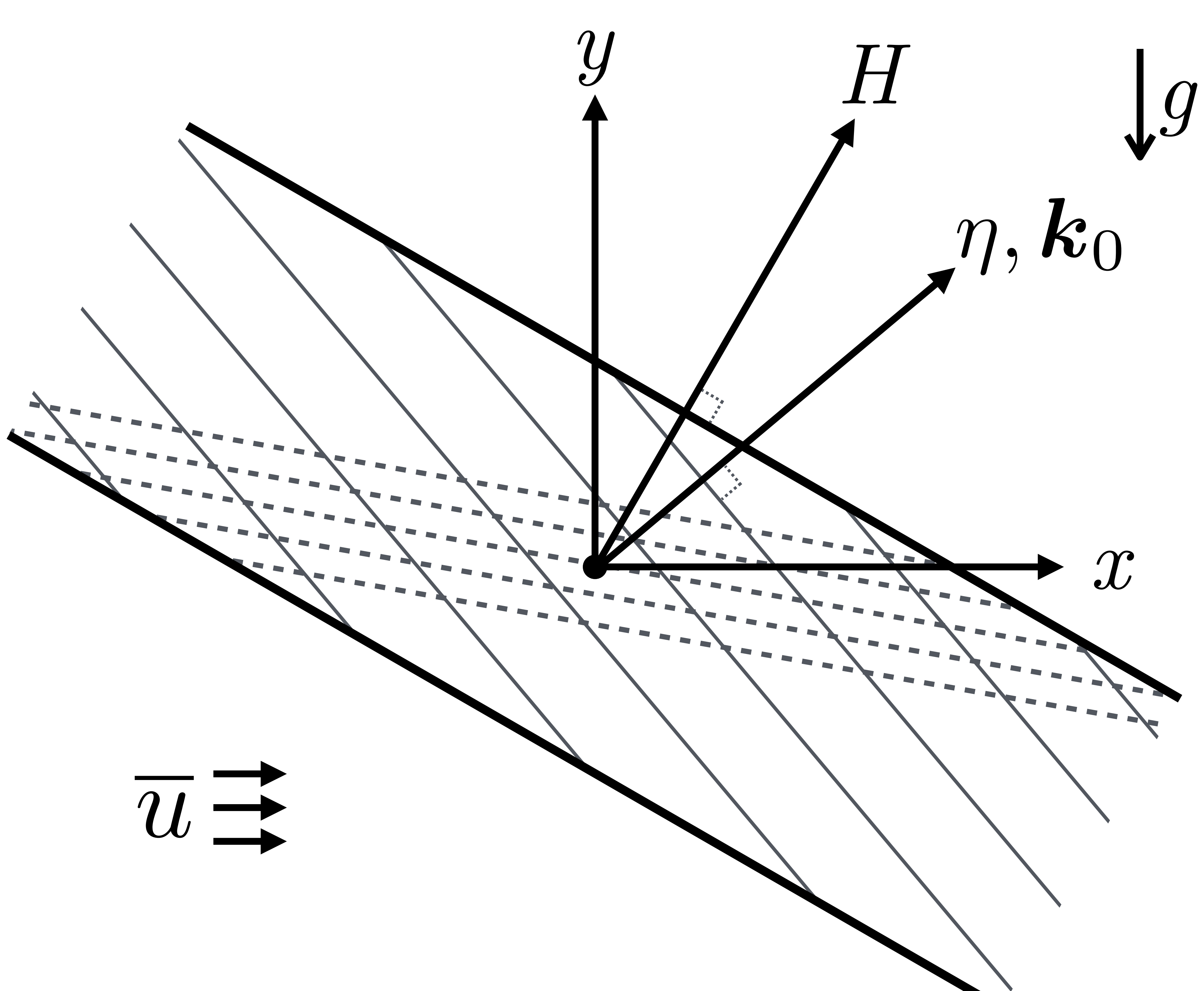
Evolution of subharmonics with mean flow

- Therefore, superimpose perturbations onto the primary beam as follows:

$$\psi = \underbrace{\epsilon\Big\{Q(H)\mathrm{e}^{\mathrm{i}(\eta-\omega_0 t)} + \mathrm{c.c.}\Big\}}_{\psi_0} + \frac{\epsilon^{3/2}}{\kappa}\Big\{A(H,T)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_+\cdot\boldsymbol{x}-\omega_+ t)} + B(H,T)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_-\cdot\boldsymbol{x}-\omega_- t)} + \mathrm{c.c.}\Big\}$$

$$\rho = \underbrace{\epsilon\Big\{Q(H)\mathrm{e}^{\mathrm{i}(\eta-\omega_0 t)} + \mathrm{c.c.}\Big\}}_{\rho_0} + \epsilon\Big\{F(H,T)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_+\cdot\boldsymbol{x}-\omega_+ t)} + G(H,T)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_-\cdot\boldsymbol{x}-\omega_- t)} + \mathrm{c.c.}\Big\}$$

perturbation envelopes
(modulated by primary wave)



Evolution of subharmonics with mean flow

- Inserting into governing equations, evolution equation for perturbation envelopes:

$$
\begin{aligned}
A_T + \frac{1}{D}\Big(\frac{c}{\kappa} + \overline{u}\sin\theta\Big)A_H + \mathrm{i}\frac{c'}{8\kappa^2}A + \alpha\kappa^2 A - \mathrm{i}\kappa^2\delta|Q|^2 A - \gamma Q B^* = 0, \\
B_T - \frac{1}{D}\Big(\frac{c}{\kappa} - \overline{u}\sin\theta\Big)B_H + \mathrm{i}\frac{c'}{8\kappa^2}B + \alpha\kappa^2 B - \mathrm{i}\kappa^2\delta|Q|^2 B - \gamma Q A^* = 0,
\end{aligned}
$$

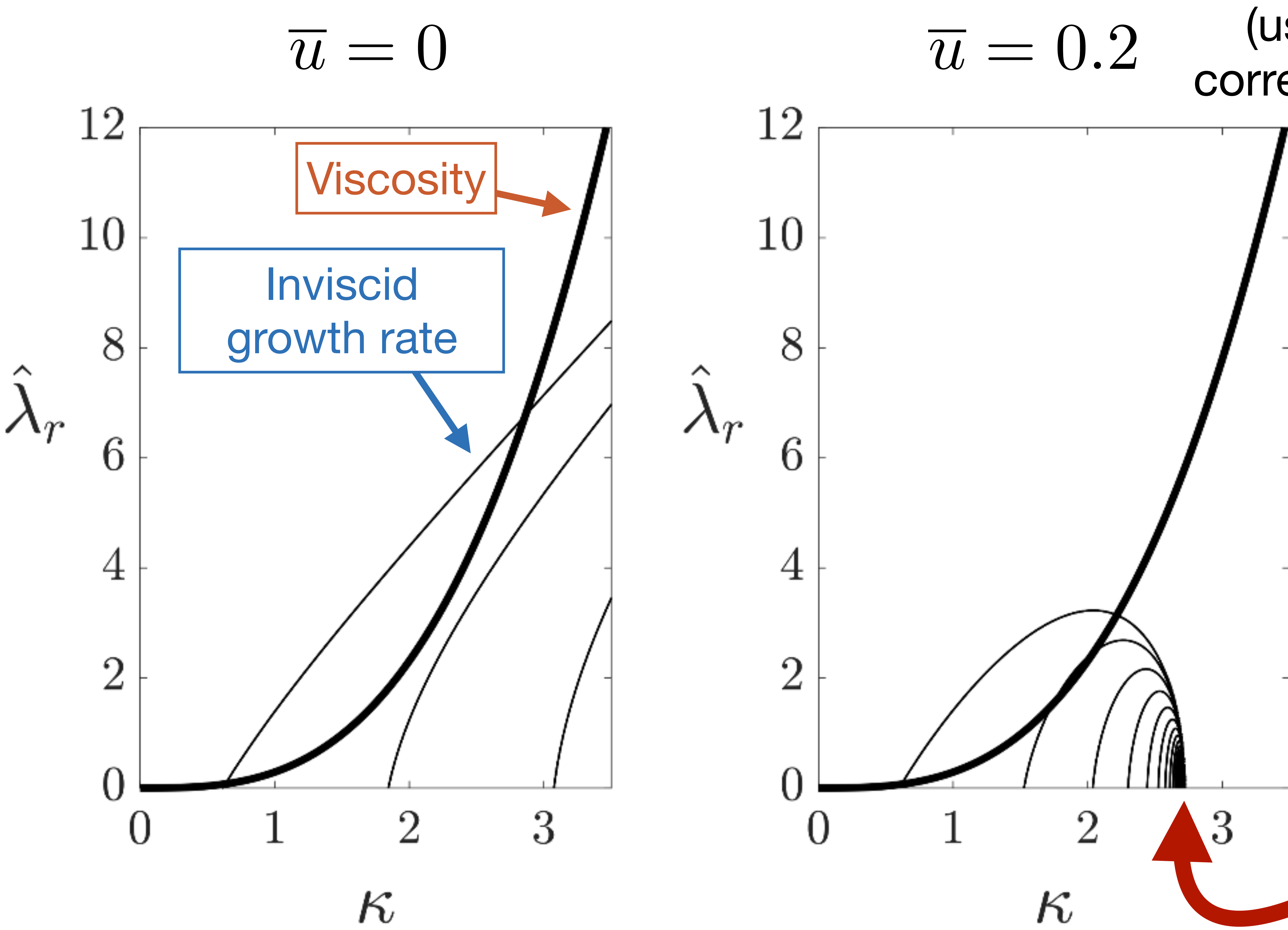
group velocity | disperson | viscous dissipation | nonlinear refraction | nonlinear energy transfer

- Envelope width D brings out **effect of finite width** — for no mean flow, return to Karimi & Akylas (2014)
- Leading order effect of mean flow **modifies group velocity** of subharmonics
- Mean flow **breaks symmetry** of left-going/right-going perturbations!

What is the effect on **PSI growth rates**?

PSI with mean flow – linear stability analysis

- Assume normal mode solutions $(A, B^*) = (a, b^*)\mathrm{e}^{\lambda T}$
- For the top-hat profile, $Q = \begin{cases} 1/2, & |H| \leq 1/2 \\ 0, & |H| > 1/2 \end{cases}$, we can derive an analytic solution
- Using $\theta = \pi/4, \quad \epsilon = 0.3, \quad \nu = 0.004, \quad D = 5$ (laboratory conditions, corresponds to wavelength ~10 cm)



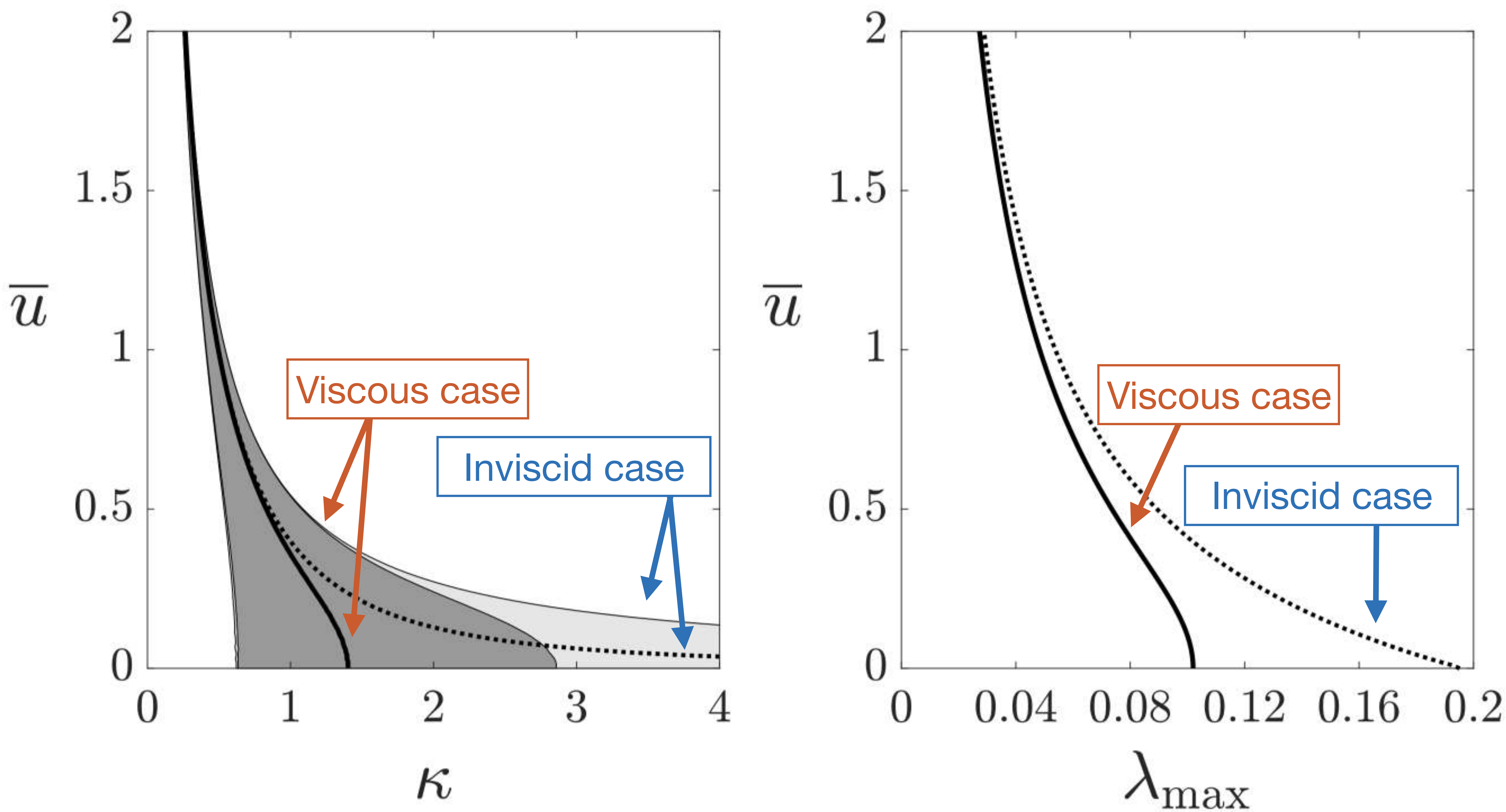
(using values above, corresponds to ~1 mm/s)

Mean flow sets necessary condition for PSI
(stabilizes high wavenumbers)

$$\kappa < \frac{c}{\overline{u}\sin\theta}$$

(subharmonics must propagate in opposite directions)

PSI growth rates for varying mean flow



Increasing background mean flow **shrinks window of instability** and **reduces growth rate**!

Nearly monochromatic PSI with mean flow – summary & conclusions

- **Weakly nonlinear** interaction between an **nearly monochromatic** internal wave beam and **fine-scale** subharmonic perturbations (PSI) with **small background mean flow**

- Mean flow introduces **max wavenumber limit** for instability (i.e. high wavenumbers are stabilized)

- Mean flow **reduces PSI growth rates** and shrinks window of unstable wavenumbers

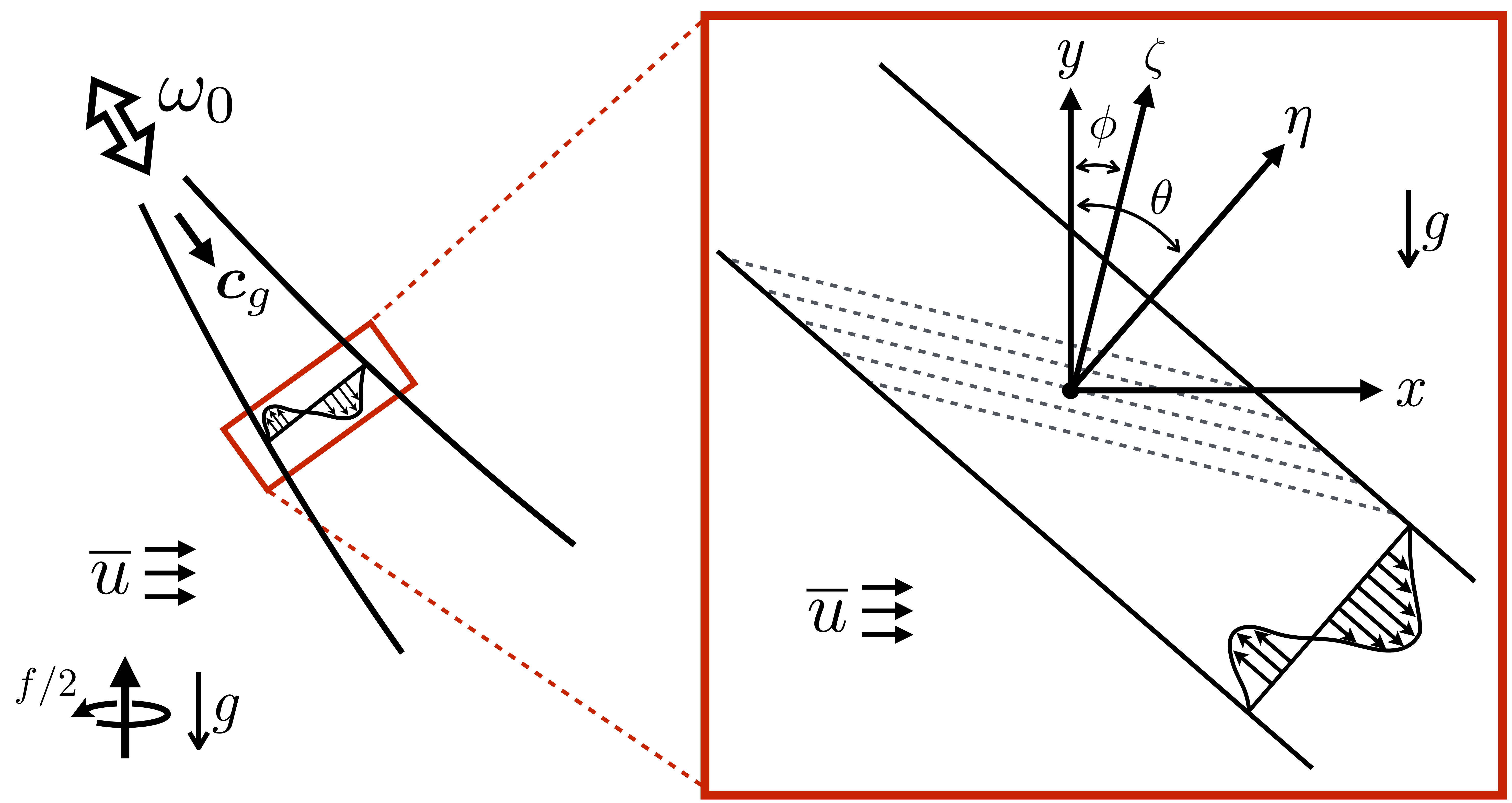
- Published in JFM (Fan & Akylas 2019)

Near-inertial PSI with mean flow

- With background rotation, beams with general profile can be unstable to PSI
- Requires that **primary wave frequency ~ 2*f***
- Perturbations at half the primary frequency are **near-inertial** — **vanishing group velocity**!
- For **small mean flow**, effect of mean flow on primary beam is small — take **slice model** to analyze effect of mean flow on evolution of perturbations

Dispersion relation

$$(\omega_0 - \overline{u}|\boldsymbol{k}|\sin\theta)^2 = \sin^2\theta$$



Near-inertial PSI with mean flow — formulation

	Frequency	**Wavenumber**	**Ansatz**
Primary beam	$\omega_0 = 2f + \sigma\epsilon$	$\boldsymbol{k}_0 = O(1)$	$\psi_0 = \epsilon\{Q(\eta)\mathrm{e}^{-\mathrm{i}\omega_0 t} + \mathrm{c.c.}\}$
Perturbations	$\omega_\pm = \frac{\omega_0}{2}$	$\boldsymbol{k}_\pm = \pm\frac{\kappa}{\epsilon^{1/2}}\hat{\boldsymbol{e}}_y + O(1)$	$\begin{cases}\psi_+ = A(x,y,T)\mathrm{e}^{\mathrm{i}\kappa y/\epsilon^{1/2}}\mathrm{e}^{-\mathrm{i}\omega_0 t/2} + \mathrm{c.c.}\\ \psi_- = B(x,y,T)\mathrm{e}^{-\mathrm{i}\kappa y/\epsilon^{1/2}}\mathrm{e}^{-\mathrm{i}\omega_0 t/2} + \mathrm{c.c.}\end{cases}$

Detuning (→ σ)

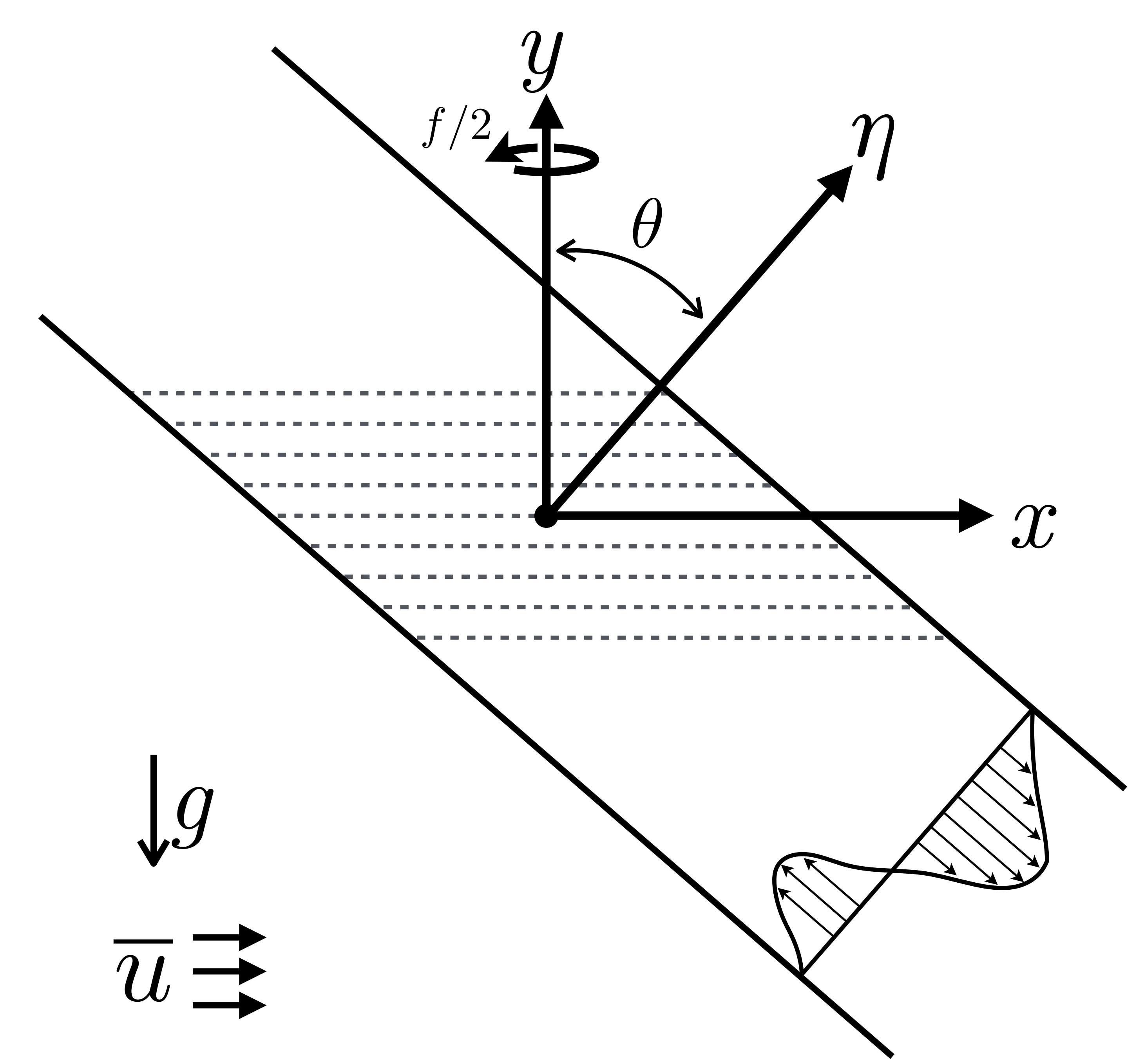
Wavenumber parameter (→ κ)

- Derive evolution equations in the **distinguished limit** — balance between effects of mean flow, weak nonlinearity, dispersion, and viscosity

$$A_T + (\alpha\kappa^2 - \mathrm{i}\frac{\sigma}{2})A + \overline{u}\sin\theta A_\eta - \mathrm{i}\frac{c'}{2\kappa^2}A_{\eta\eta} - \mathrm{i}\kappa^2\delta|Q_\eta|^2 A + \gamma Q_{\eta\eta}B^* = 0$$

$$B^*_T + (\alpha\kappa^2 + \mathrm{i}\frac{\sigma}{2})B^* + \overline{u}\sin\theta B^*_\eta + \mathrm{i}\frac{c'}{2\kappa^2}B^*_{\eta\eta} + \mathrm{i}\kappa^2\delta|Q_\eta|^2 B^* + \gamma Q^*_{\eta\eta}A = 0$$

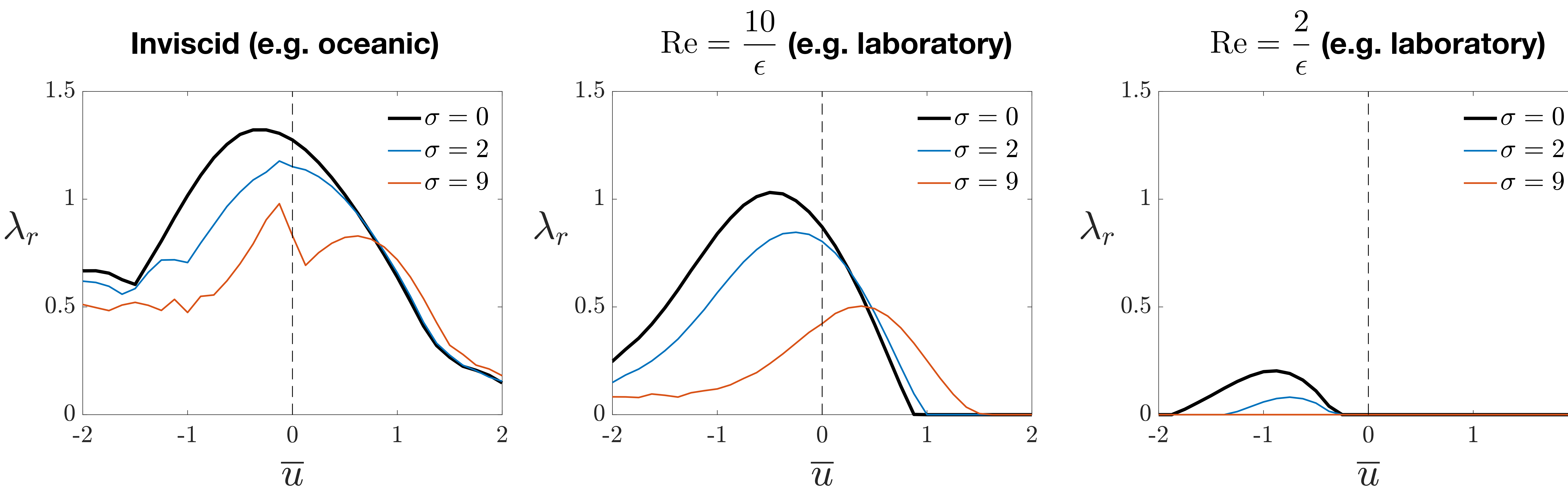
Viscous dissipation | **Detuning** | **Mean flow advection** | **Dispersion** | **Nonlinear refraction** | **Nonlinear energy transfer**



Near-inertial PSI with mean flow – preliminary results

- Performing a **linear stability analysis** using **locally confined (Gaussian-like) profile**, PSI growth rates are shown:

Primary wave beam frequency $\omega_0 = 2f + \epsilon\sigma$



ϵ is nondimensional amplitude (typically ~0.1 in laboratory)

Concluding remarks

- In the weakly nonlinear limit, presence of **background mean flow weakens PSI**
- **Without background rotation**, mean flow introduces additional necessary condition for instability for nearly monochromatic waves and **reduces PSI growth rates** (Fan & Akylas 2019)
- **With background rotation**, mean flow also reduces **PSI growth rates** (Fan & Akylas, in preparation)
    - Situation is complicated when allowing for sub-inertial perturbations ($\omega_0/2 < f$) and is currently under investigation